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ON DE-BUNKING ARITHMETIC.

By M. M. Rogers.

They wept like anything to see
Such quantities of sand.
"If this were only cleared away,"
They said, "it would be grand."

It would be; and if those young airmen of whom we heard in the February *Gazette* had not spent so much lost endeavour in weaving ropes of this sand, perhaps they would not now be in such need of the help which we are so eager to give them. For although the seven-times-seven mops of our teaching committees have been sweeping for far more than half a year, there is no doubt that school Arithmetic is still cumbered with a lot of stuff that is of no use outside the class-room. In spite of the fact that teachers mostly teach so well, and pupils learn so well, we must confess that children going out into the world often cannot do the calculations that they need, that Training College lecturers find that many students have to learn Arithmetic all over again (though we camouflage it as "Method"), and that intelligent well-educated folk seem to delight in telling anyone whom they suspect of mathematical leanings that "they never could do Arithmetic". Surely it is the Arithmetic that is wrong, not the teachers or taught.

For many years, I have been trying to find out what the trouble is. I have consulted teachers informally, watched countless Arithmetic lessons from the position of the onlooker who sees most of the game, tried to help the students who gave those lessons by seeking the cause of their difficulties, and finally invited the students themselves to join in the hunt, which they did with great glee. We found that among smaller children at any rate the alleged hatred of Arithmetic is often a myth. There was every degree of enjoyment, from the small imps who were deliberately naughty in order to get sums for a punishment (we had more than one instance of this kind) to the thirteen-year-old boys delighting in differential calculus, to the great amazement of the inspector.* But we found two main causes of trouble: one was the confusion caused by diversity of method, the other was the useless burden that our poor subject still has to carry. The first of these is a grievous stumbling-block; its seriousness was accentuated by the re-organisation of schools, and is now increased ten-fold by the problems of evacuation. For the sake of the children, unanimity is sorely needed in such matters as (*a*) subtraction, (*b*) long multiplication, (*c*) the determination of the decimal point. We are almost agreed as to (*a*). We agree in theory as to (*b*); our own Report, the Board of Education Handbook, and most textbooks recommend that multiplication should begin with the leading digit. But the diversity of practice that still obtains is shown in Mr. Webb's investigation,† where

* M. V. Hughes, *A London family between the wars.*

† *Mathematical Gazette*, October 1940.

different methods appear on successive pages. Pupils often begin with the unit digit when multiplying integers, and reverse the order when a decimal point appears. (*c*) seems to be still in the melting-pot, unfortunately for our children.

However, this paper is chiefly concerned with the second difficulty. It is true that there has been much improvement; it is true that our critics often seem unaware of this, and belabour a past that we hope is dead. But school Arithmetic lags a bit behind our desires, perhaps because its pastors and masters do not always read the publications of the Mathematical Association. Even good modern textbooks are apt to suffer from two opposite dangers. In the desire to be practical, the fact that Arithmetic is the science of pure number may be obscured. A good Infant School teacher once said, "Surely the children ought to know not only that 5 apples and 6 apples make 11 apples, but that 5 and 6 are 11", and I, for one, believe that many children prefer to have it so, that they will do sheer calculation for the fun of it, rightly gloating over the row of ticks on their pages that proclaim perfection, and that if one has to add 345 to 276 it is a hindrance not a help to label them pigs or nuts, neither of which would ever be counted in such numbers. (This is an actual example; I could not have invented it).

On the other hand, the would-be practical stuff is not practical enough. Life encounters situations, and needs Arithmetic to meet them; textbooks like to propound a rule, and then, with much ingenuity, invent situations to illustrate it, thereby producing what the students joyously christened "sham real" examples. The glorious job of de-bunking Arithmetic would clear away what still remains of sham reality, making room for much better number work, applied to such matters and in such ways as really need it. Three clearance areas may be suggested, in addition to some which have already been swept out. The first and largest covers the compound rules of money, weights and measures. Herbert McKay's *Odd Numbers* * has an interesting chapter on the subject which gives a somewhat unexpected defence of the English system against the decimal, on the grounds that, characteristically enough, it works out far better in practice, and I have heard an able engineer advance the same argument. McKay says pertinently, "Decimalists would have us abolish the convenience because of the absurdity; it seems more sensible to abolish the absurdity and keep the convenience". He says that most of the difficulty of calculation does not matter, because "outside Arithmetic books such things do not happen". He then quotes an example which we hope no longer happens inside Arithmetic books, as his reviewer in the *Gazette* † has pointed out. But a good deal does still happen there to which his strictures apply; and we could save a year's work at a vital point in the school career if we were to eliminate from the pages of our books all supposedly

* Herbert McKay, *Odd Numbers* (C.U.P. 1940).

† *Mathematical Gazette*, October 1940.

concrete examples which do not occur outside those pages. Here are some suggestions made on this principle—does it happen? The examples quoted are not extreme ones, and are taken from recent books of good standing.

With regard to money sums: first lessen the amount of attention given to farthings. Their actual use is limited to a few specific cases, chiefly the something-and-eleven-three of the drapers. Many districts do not use them at all, they are unknown in Woolworth's, where the small people who wrestle with them in school enter the world of real finance, and the time and effort spent on them in class seems out of all proportion to the need. We still have pages and pages of additions every item of which brings in farthings, followed by multiplications like $2/8\frac{3}{4} \times 7$. It is almost impossible to invent an actual bill to fit the first; and what about the second? The $/11\frac{3}{4}$ of course is treated as the next shilling; the necessary adjustment would be the most useful exercise required in this connection.

The next suggestion seems impertinent in view of the interesting discussions recently published in this *Gazette*. But—does long multiplication of money really happen? If there is such a sum as £156 13s. 8¾d., could there ever be 347 of them? This question is not rhetorical—it is very far from being sarcastic in intention—but has anybody ever had to do such a sum outside school? and if not, must we labour so woundily at it in school?

Next, apply the principle of "does it happen?" to the rest of the compound rules. For long enough, we have limited these to three consecutive units, instead of the miles-to-inches of Mr. McKay's bugbear. But even within these limits, there are still pages and pages which could well be spared. We still find:

Add	hrs.	min.	secs.	yds.	ft.	in.
	2	46	26	3	1	11
	4	53	31	1	1	8
	1	11	43	2	2	10

Subtract	wks.	dys.	hrs.
	7	3	12
	3	4	19

Multiply 3 yds. 2 ft. 11 in. by 17.

How many times is 5 hrs. 49 min. contained in one week?

It is true that odd weights do have to be added by those who deal with lading, odd gallons quarts and pints by milkmen. But it is difficult to imagine cases in which the odd lengths quoted above, still less the times, could be added or subtracted, and there seems no sense in dividing 61 minutes 23 secs. by 29, or in multiplying quantities of which multiples could not exist. Surely £57 9s. 4¾d. has no meaning as an aggregate of farthings, and it is of doubtful benefit to change or convert quantities from one measure to another by which they never would have to be reckoned. We could probably

do without 80 or 90 per cent. of the examples on Compound rules; yet not long ago I saw a whole book in a Revision series devoted to them alone.

My second clearance area is that of the Unitary Method. From time to time in the pages of this *Gazette* its august position is impugned; the presidential address a year or two ago had some refreshing remarks on the subject. The recent report of the Metropolitan branch of the Training College Association * says that "it is only useful in simple cases and is not of general application". Yet in schools and in textbooks it appears to be sacrosanct; I have come across teachers who seem to regard it as the philosopher's stone, and one wonders why. As far as my own experience goes, I have found again and again that dependence on the unit line completely stultifies all sense of proportion, that so far from paving the way to the fascinating conception of variation in all its forms, it provides a heavy stumbling-block. If we apply the principle of "does it happen?" the result is obvious, as anybody can see who tries to scrape together a set of examples with the barest veneer of speciousness. Even where the supposed uniformity of variation exists—often it does not—it is difficult to imagine cases in which such data would be given to determine such results; e.g. the cost of $3\frac{1}{2}$ yards is usually determined from the price per yard, not strangely deduced from that of $2\frac{3}{4}$ yards. I am not sure whether we still embark on such problems as n men working m hours a day building a wall p feet long . . . etc., etc.; but in my unregenerate youth we used to put the figures where they would cancel and hope for the best.

It is true that many good teachers consider that the principle of "finding one first" is a good introduction, and probably it is worth retaining if it is used as such and shown to lead to the better conception. But even in the early stages, my own inclination is to prefer the idea of "how many times as much?" avoiding premature use of the word *ratio*, which seems a bit terrifying to the non-specialist.

Thirdly "floors have to be carpeted and walls papered under conditions that never exist in this world"; † accordingly so-called area and volume examples furnish our next sham-real clearance area. Wall-papering and carpet-measuring problems are introduced as exercises in area, but the householder finds that they are far more often a matter of fitting in two linear dimensions, and that he—or more probably she—is not in the least concerned with the resulting area. For example, a textbook asks us to find what length of carpet is needed to cover a floor 14 ft. long and 8 ft. 6 in. wide. If one tries to think of a real carpet, which is the last thing the children would probably do, the first idea would be, "What an odd room". Still more odd will it be to put the carpet right up to the walls; odder still, to make the strips go the wrong way, just because 21″ is contained in 14′ and not in 8′ 6″. When we have done all this,

* *Arithmetic in Junior Schools* (Longmans), 1940.

† *Mathematical Gazette*, July 1940, p. 191.

we should still find it hard to persuade the shop to sell us the 68 feet of carpet required by the answer.

Another book has lovely plans of a house to be furnished all through. The children love this, and here we surely will find real carpets. But what happens? We are asked to decide on the width of border in each room, then find the area of the resulting carpet. Anybody who buys a carpet knows that carpets of these sizes do not exist, nor could they be made from the standard widths. Sham-reality has turned the problem upside-down; it is the standard size of possible carpets that determines the width of the border, not *vice versa*. Anyhow, one is not much concerned with the area, if at all. The same sort of thing is true of wall-papering, where the real problem is to see how many strips it takes to go round the room, then how many complete strips can be cut from each roll of paper. Dividing the area of the walls by the area of a roll is actually wrong, since paper cannot be poured on walls like paint.

Here is a "volume" example from an examination set last year, concerning the amount of water contained in a bucket. Its dimensions were given, then followed the instruction, "Take 10 gallons to a cubic foot". The candidate said afterwards, "I thought I must be wrong; I knew a bucket could not hold as much as that". She could easily have tackled an abstract example on a truncated cone, but why call it a bucket of water and then make it impossible for it to be anything of the kind by adding a statement which is gratuitously false?

All this may give rise to the objection that our attitude towards Mathematics should not be too utilitarian. Agreed, and may all the gods forbid that anybody who writes in these pages should seek to degrade the Queen of sciences to the office of a kitchenmaid. But the kind of stuff that we want to get rid of is not good Mathematics any more than it it is good sense; we want it out of the way in order to make room for better Mathematics as well as better sense. Also, in the endeavour to simplify working, I do not want to forget the factor-of-safety; on the other hand, I should like to thank Mr. Tuckey for his defence of it* (as well as for the whole of his paper, which said what we all want said far better than most of us could say it for ourselves). But, if I may borrow his metaphor, Jack ensures his leap across the ditch by practising long jumping, not high jumping or putting the weight. Least of all do I want to indulge in the cheap sport of knocking down without trying to build up. Any adequate attempt to do this would double the length of this paper, and I have tried to make fuller suggestions elsewhere.† In brief, the first thing would be to get real proficiency in the mere elements of calculation. The necessary knowledge is so little, it would carry us so far, but indeed and alas! so few people possess it. It could be such a pleasant task to acquire an intelligent and

* *Mathematical Gazette*, July 1940.

† M. M. Rogers, *Arithmetic is Easy* (Evans, 1940).

accurate measure of skill, as in a kind of numerical " keep-fit " class. The next aim would be to collect from employers the kind of things that they wish their young people knew, also from the same young people the kind of things that they wish we had taught them, and to frame the course accordingly. Probably the actual Arithmetic involved would be considerably less than the traditional syllabus, which is greatly to be desired, as we could then include other branches of Mathematics, as indeed, many of the better books do already. Two years ago a correspondence in the daily press gave an excellent opportunity for such a procedure; but the events of that September intervened, and have kept most of us otherwise occupied ever since.

One point remains, perhaps the most important. Last year, I had the pleasure of being present at a conference of head teachers where these matters were discussed. There was much agreement in principle, but two pertinent remarks were made at the end. One said, " All this is true, but we feel that we have a workable tradition, at the end of which children *can do* Arithmetic " (herein I fear he was too optimistic), " and we are not prepared to abandon it until we are sure that we have something better ". Another said, " The change must come from above. As long as Inspectors ask and examinations set questions of the traditional kind, we *must* prepare our children for them ". Of course, this is so. The Board of Education * still includes addition, subtraction, short multiplication and division of weights, measures *and time* in the syllabus for Junior schools, long multiplication and division of them all in the Senior syllabus. The recent report issued by the Mathematical section of the Metropolitan branch of the Training College Association † is much more merciful. But textbooks do still print and examiners do still set the kind of stuff that all our teaching committees deplore, and as long as they do, teachers (poor things) must go on teaching it. How can we speak with enough authority? Could the Board of Education do it, calling to its aid the myriad voices of all the wireless sets in the land, to proclaim that teachers really are freed from these burdens? The schools now are working under great difficulties. They have had to give up many precious things; surely it would help them if we could seize this opportunity to free them for ever from some bad things. To do so would be one of the many ways in which good is being brought out of the evil of these strange times.

M. M. R.

GLEANINGS FAR AND NEAR.

1371. Peter [the Great, of Russia] even went so far as to prohibit any nobleman from marrying until he had passed an examination in geometry, arithmetic and navigation.—Arnold J. Toynbee, *A Study of History* (Oxford, 1934), iii, p. 283, foot-note.

* *Educational Pamphlets*, No. 101. Senior School Mathematics.

† *Arithmetic in Junior Schools.*

TEACHING THE CALCULUS.*

By R. L. Goodstein.

One of the fundamental difficulties in teaching the calculus is that we are obliged to introduce the *concepts* of the Theory of Functions and yet make no use of the real number theory on which the Theory of Functions is based. We must talk of limits, continuous functions, derivatives, definite integrals and of existence theorems like Rolle's and Taylor's, and must endeavour to hide the deficiencies of our reasoning beneath the kindly cloak of geometrical intuition. The Calculus course is usually a compromise, completely rigorous proofs of the basic theorems being deemed too difficult for the type of student for whom the course is planned, but I believe a far more rigorous account than the customary one is possible without greatly increasing the strain on the students' powers of understanding; this can be achieved, however, only by a new approach to the subject and it is with the first steps in such an approach that we are here concerned.

We shall consider three theorems.

(1) *A convergent sequence tends to a limit.*

(2) *A continuous function takes any value between any two of its values.*

(3) *A continuous function $f(x)$ has a definite integral $F(x)$, the derivative of which is $f(x)$.*

It is important to emphasize that we do not pretend in this brief account to use the informal language in which the ideas should be dressed for teaching purposes.

We start by talking of the idea of a decimal fraction and distinguish finite decimals, recurring decimals and non-recurring endless decimals. Next we consider decimals evaluated to so many places and observe that all the familiar functions, $\sin x$, $\cos x$, $\log x$ are known to us from table-books to four or five or seven decimal places, and prepare the student to accept the idea of a number n as an endless decimal and of the number $(n)_r$ consisting of the first r decimal digits of n. Illustrate by such examples as: $\sqrt{2} = 1{\cdot}4142\ldots$, $(\sqrt{2})_3 = 1{\cdot}414$, $(\log_{10} 2)_5 = {\cdot}30103$, $(3/64)_3 = {\cdot}046$, $a = {\cdot}00101010001\ldots$, the nth digit being 1 or 0 according as n is an odd prime or not, $(a)_5 = {\cdot}00101$, $(a)_{31} = (a)_{36}$.

Next we turn our attention to sequences of decimals. What do we observe in the sequence $a_1 = {\cdot}31$, $a_2 = {\cdot}332$, $a_3 = {\cdot}3333$, $a_4 = {\cdot}33334$, $a_5 = {\cdot}333335, \ldots a_{26} = {\cdot}33\ldots326, \ldots$?

We notice that $(a_n)_1$ is the same for all n, being ·3, that $(a_n)_2$ is the same for all $n \geqslant 2$, being ·33, and that generally $(a_n)_r$ is the same for all $n \geqslant r$, i.e. the greater the value of n the more digits a_n has in common with the terms that follow it. The characteristic property of the numbers $(a_1)_1$, $(a_2)_2$, $(a_3)_3$, ... is expressed by the equations

* The remarks in this article are intended to apply to university teaching. *See* the final paragraph.

$(a_r)_r=(a_n)_r=((a_n)_n)_r$ for any $n\geqslant r$. Let l be the decimal whose nth digit, for each n, is the nth digit of a_n; then $(l)_r=(a_r)_r=(a_n)_r$ for any $n\geqslant r$, i.e. l has as many digits as we please in common with a_n for a sufficiently great n (in fact $l=\cdot333\ldots=\cdot\dot{3}$).

A sequence $u_1, u_2, u_3, \ldots$, such that for each r, $(u_m)_r$ is *constant* for *all* sufficiently great values of m, is said to be convergent and the number l whose nth digit is the nth digit of the constant value of $(u_m)_n$ is called the limit of the sequence. A convergent sequence is said to converge to its limit. If m_r is the first value of m for which $(u_m)_r$ takes its constant value, and if we pick out the terms $u_{m_1}, u_{m_2}, u_{m_3}, \ldots$ and write them for brevity as $v_1, v_2, v_3 \ldots$, then the condition satisfied by any v_r is that $(v_r)_r=(u_n)_r$ for any $n\geqslant m_r$, i.e. the first r digits of v_r are common to all u_n for $n\geqslant m_r$. The limit l, by definition, satisfies $(l)_n=(v_n)_n=(u_m)_n$, $m\geqslant m_n$.

For practical purposes it is customary to express the convergence condition rather differently. We say that a sequence $u_1, u_2, u_3, \ldots$ is convergent if we can find a sequence $n_1, n_2, n_3, \ldots$ such that for any r and $n\geqslant n_r$, $|u_n-u_{n_r}|_{r+1}=0$. In general this condition is equivalent to the foregoing since it is generally true that if u_n differs from u_{n_r} by less than $1/10^{r+1}$ then $(u_n)_r=(u_{n_r})_r$ and the limit of the sequence is determined as before; the exceptional case is exemplified by such a pair of numbers as $u_n=2\cdot731999\ldots$, $u_{n_3}=2\cdot732$ for which $u_{n_3}-u_n\leqslant 1/10^6$ but $(u_n)_3$ is not equal to $(u_{n_3})_3$. In such a case three possibilities may arise:

(i) We can find a k such that $(u_{n_k})_k$ is greater than $2\cdot732$.

(ii) We can find a k such that $(u_{n_k})_k$ is less than $(2\cdot731\dot{9})_k$.

(iii) There is no k for which $(u_{n_k})_k$ is greater than $2\cdot732$ and no k for which $(u_{n_k})_k$ is less than $(2\cdot731\dot{9})_k$.

If (iii) occurs, every $(u_{n_k})_k$ is either $2\cdot732$ or $(2\cdot731\dot{9})_k$, and since $2\cdot731\dot{9}=2\cdot732$ we take $2\cdot732$ to be the limit of the sequence.

In (i) all u_n, from u_{n_k} onwards, are not less than $2\cdot732$ (because $(2\cdot732+1/10^k)-2\cdot732=1/10^k$ and $u_{n_k}-u_n<1/10^{k+1}$ for $n\geqslant n_k$) and the limit l of the sequence satisfies $(l)_3=2\cdot732$, for $(u_n)_3=2\cdot732$ for all $n\geqslant n_k$.

In (ii) all u_n, from u_{n_k} onwards, are less than $(2\cdot731\dot{9})_k$ (because $(2\cdot731\dot{9})_k-((2\cdot731\dot{9})_k-1/10^k)=1/10^k>1/10^{k+1}$) and accordingly $(l)_3=2\cdot731$.

Thus, whichever possibility occurs, the digits of the limit of a sequence $u_1, u_2, u_3, \ldots$ which satisfies $(u_n-u_{n_r})_r=0$ for $n\geqslant n_r$, are determined one by one and in every case the limit l satisfies the condition $(l-u_n)_r=0$ for $n\geqslant n_r$.

The first possibility we mentioned is illustrated by the sequence 2, 2·7, 2·73, 2·732, 2·731$\dot{9}$, 2·731999$\dot{2}$, 2·732001, 2·7320011, 2·73200111, etc., in which from the 7th term onwards the integral part and first three decimal places are constant and form 2·732.

The second possibility is illustrated by the sequence 2, 2·7, 2·73,

2·732, 2·7319, 2·73199, 2·73200, 2·731994, 2·7319944, 2·73199444, ..., etc., in which from the 8th term onwards the integral part and first three decimal places are constant and form 2·731.

The third possibility is illustrated by the sequence 2, 2·7, 2·73, 2·732, $2{\cdot}7319\dot{6}$, 2·732, $2{\cdot}731994$, 2·732, $2{\cdot}7319996$, 2·732, $2{\cdot}73199994$, ..., etc., of which the limit is 2·732 or $2{\cdot}731\dot{9}$.

If a sequence $u_1, u_2, u_3, \ldots$ and a number l satisfy the condition $(l-u_n)_r=0$ for $n \geqslant n_r$, then the sequence u_n is convergent and l is the limit of the sequence. For, if $n \geqslant n_{r+1}$, $|u_n-u_{n_{r+1}}|=|(l-u_{n_{r+1}})-(l-u_n)| \leqslant |l-u_{n_{r+1}}|+|l-u_n| \leqslant 2/10^{r+1}$ and so $(u_n-u_{n_{r+1}})_r=0$; thus $u_1, u_2, u_3 \ldots$ is convergent and if λ is the limit of the sequence then $(\lambda-u_n)_r=0$, $n \geqslant n_{r+1}$, and so for any r, $(l-\lambda)_r=0$, i.e. corresponding digits of l and λ are the same and therefore $l=\lambda$ (we suppose that neither l nor λ is expressed with an unlimited recurrence of 9's).

In a first account of this treatment of convergence no mention should be made of what we called the exceptional case, the consideration of which should be left until the student is fully familiar with the notation and terminology. A good way of illustrating the technique is to consider such problems as the multiplication of $\sqrt{2}=1{\cdot}4142\ldots$ and $\sqrt{3}=1{\cdot}7320\ldots$. We carry out in turn the operations $1{\cdot}4 \times 1{\cdot}7 = 2{\cdot}38$, $1{\cdot}41 \times 1{\cdot}73 = 2{\cdot}4393$, $1{\cdot}414 \times 1{\cdot}732 = 2{\cdot}449048$, $1{\cdot}4142 \times 1{\cdot}7320 = 2{\cdot}4493944$; from the first product onwards the integral part is constant, from the second onwards the first decimal figure is constant, from the third onwards the second decimal figure is constant, and so on. To test how many figures of any product are common to any subsequent product, for instance to see how many figures of the last product will remain constant in future products, we might argue that $1{\cdot}4142 < \sqrt{2} < 1{\cdot}4143$, $1{\cdot}7320 < \sqrt{3} < 1{\cdot}7321$, and since $1{\cdot}7321 \times 1{\cdot}4143 = 2{\cdot}44970903$ it follows that $2{\cdot}4493\ldots < \sqrt{2} \times \sqrt{3} < 2{\cdot}4497\ldots$, and therefore to three places $\sqrt{2} \times \sqrt{3}$ (which is the limit of the sequence of products) equals 2·449. This is precisely the convergence idea; if we think of the successive developments of $\sqrt{2}$ as forming a sequence $u_1=1{\cdot}4$, $u_2=1{\cdot}41$, $u_3=1{\cdot}414, \ldots$, and the developments of $\sqrt{3}$ as forming the sequence $v_1=1{\cdot}7$, $v_2=1{\cdot}73$, $v_3=1{\cdot}732, \ldots$, then both the sequences are convergent since $(u_m-u_n)_n=0$ and $(v_m-v_n)_n=0$ for $m \geqslant n$. The product sequence $u_1v_1, u_2v_2, u_3v_3, \ldots$ is convergent since

$$\begin{aligned} |u_mv_m-u_nv_n| &= |u_mv_m-u_nv_m+u_nv_m-u_nv_n| \\ &\leqslant |v_m(u_m-u_n)|+|u_n(v_m-v_n)| \\ &< 2|u_m-u_n|+2|v_n-v_n| \\ &< 4/10^n < 10^{n-1}, \end{aligned}$$

i.e. $(u_mv_m-u_{n+1}v_{n+1})_n=0$ for $m \geqslant n+1$, and so we know in advance that more and more decimal figures will become constant as we carry out the successive multiplications $u_1v_1, u_2v_2, u_3v_3, \ldots$, and furthermore we know that the first $n-1$ decimal figures of $u_{n+1}v_{n+1}$ will be common to all succeeding products u_mv_m, $m>n+1$.

Another good illustration is afforded by considering the products $1{\cdot}1 \times 1{\cdot}3$, $1{\cdot}11 \times 1{\cdot}33$, $1{\cdot}111 \times 1{\cdot}333$, $1{\cdot}1111 \times 1{\cdot}3333$, and so on, obtaining step by step the digits of $1{\cdot}\dot{3} \times 1{\cdot}\dot{1} = 40/27$.

Continuity.

$f(x)$ is continuous in the interval (a, b) if, for each value of r, we can determine a length l_r so that for any two points, x, x' (between a and b) whose distance apart is not greater than l_r, $(f(x) - f(x'))_r = 0$.

It follows that if $f(x)$ is continuous in (a, b) it is continuous in any part of (a, b); furthermore if N_r is the first integer above $1/l_r$ then we can divide the interval (a, b) into N_r equal parts in each of which $(f(x) - f(x'))_r = 0$ for any x, x' in the same part.

Theorem 1.

If $f(x)$ is continuous in (a, b) and if the sequence of points $x_1, x_2, x_3, \ldots$ all in (a, b) converge to a limit x in (a, b), then the sequence $f(x_1), f(x_2), f(x_3), \ldots$ converges to the limit $f(x)$.

Proof. Since $(x - x_n)_r = 0$ for all $n \geqslant n_r$, it follows that if we choose r so that $1/10^r < l_p$ then $(f(x) - f(x_n))_p = 0$ for $n \geqslant n_r$ and so $f(x_1)$, $f(x_2), f(x_3), \ldots$ converges to the limit $f(x)$.

Theorem 2.

If $f(x)$ is continuous in (a, b), and if $f(a)$ is less than zero and $f(b)$ greater than zero, then we can find a point c between a and b such that $f(c) = 0$.

Proof. First we divide (a, b) into N_1 equal parts by the chain of points $a = x_0, x_1, x_2, \ldots x_{N_1} = b$, where N_1 is chosen so that $(f(x) - f(x'))_1 = 0$ for any x, x' in the same part and $N_1 > 10(b - a)$. Consider in turn the N_1 numbers $f(x_0), f(x_1), f(x_2), \ldots f(x_{N_1})$. The first of these numbers is less than zero and the last positive; there must therefore be a first which is greater than or equal to zero. Let this be $f(x_{k+1})$, then $f(x_k)$ is negative and $f(x_{k+1})$ non-negative and since $(f(x_{k+1}) - f(x_k))_1 = 0$, therefore $(f(x_{k+1}))_1 = 0$. Denote the interval (x_k, x_{k+1}) by I_1. Divide I_1 into N_2 equal parts by the chain of points $x_k = x_0', x_1', x_2', \ldots x_{N_2}' = x_{k+1}$ so that $(f(x) - f(x'))_2 = 0$ for any x, x' in the same part, and $N_2 > 10$. The N_2 numbers $f(x_0')$, $f(x_1')$, $f(x_2'), \ldots f(x_{N_2}')$ commence with a negative number and terminate with a non-negative number so that there must be a first $f(x_{l+1}')$, say, which is non-negative. Since $(f(x_{l+1}') - f(x_l'))_2 = 0$ and $f(x_l')$ is negative, it follows that $(f(x_{l+1}'))_2 = 0$.

Denote the interval (x_l', x_{l+1}') by I_2. Divide I_2 into N_3 equal parts in each of which $(f(x) - f(x'))_3 = 0$, with $N_3 > 10$; choose the part I_3 as before and continue the process. If we use the letters a_k, b_k to denote the left and right hand end-points of the interval I_k, then since by definition I_{k+1} is less than a tenth part of I_k, we have $a_k \leqslant a_{k+1} < b_{k+1} \leqslant b_k$ and $b_k - a_k < 1/10^k$, whence $(a_n - a_k)_k = 0$, $(b_k - b_n)_k = 0$ for $n \geqslant k$ so that both $a_1, a_2, a_3 \ldots$ and $b_1, b_2, b_3, \ldots$ are convergent. Furthermore, for each r, $(f(b_r))_r = 0$. If c is the limit

of the convergent sequence $b_1, b_2, b_3, \ldots$ then c is contained in the interval (a, b) and so by Theorem 1, for every n, $(f(c)-f(b_k)_{n+1}=0$ for $k \geqslant$ some k_n. Thus

$$|f(c)| \leqslant |f(b_k)| + 1/10^{n+1} < 1/10^k + 1/10^{n+1} < 2/10^{n+1},$$

i.e. $(f(c))_n = 0$ for any n, so that every decimal figure of $f(c)$ is zero and therefore $f(c)=0$.

The definite integral.

The area of the rectangle formed by the line $y=k$, the x-axis, and the ordinates $x=a$, $x=b$ is called the *definite integral* of the constant function k from a to b and is denoted by the sign $\int_a^b k\,dx$. If $f(x)$ takes the constant values $k_0, k_1, k_2, k_3, \ldots k_n$ in the intervals $(a, a_1), (a_1, a_2), (a_2, a_3), \ldots (a_n, b)$, then we define

$$\int_a^b f(x)\,dx = \int_a^{a_1} k_0\,dx + \int_{a_1}^{a_2} k_1\,dx + \int_{a_1}^{a_3} k_2\,dx + \ldots + \int_{a_n}^b k_n\,dx.$$

We show next how to derive the concept of the definite integral of a general function, one which is not constant in stretches. Consider for example the function $\sqrt{x}$; the function $\sqrt{x}$ is not constant in any interval however small, but the function $(\sqrt{x})_p$, for each value of p, takes only a finite number of values in any (finite) interval; for any x in the interval $\{N^2/10^{2p}, (N^2+2N)/10^{2p}\}$, $(\sqrt{x})_p$ takes the constant value $N/10^p$, and so we can divide any interval (a, b) into a finite number of intervals

$$(a, a_1), (a_1, a_2), (a_2, a_3), \ldots (a_n, b), \text{ say,}$$

such that, for each r, $(\sqrt{x})_p$ takes a constant value k_r for $a_r \leqslant x < a_{r+1}$, and therefore $\int_a^b (\sqrt{x})_p\,dx = k_0(a_1-a) + k_1(a_2-a_1) + \ldots + k_n(b-a_n)$.

We can show that the sequence

$$\int_a^b (\sqrt{x})_1\,dx, \quad \int_a^b (\sqrt{x})_2\,dx, \quad \int_a^b (\sqrt{x})_3\,dx, \ldots$$

is convergent; the limit of the sequence defines $\int_a^b \sqrt{x}\,dx$. Since

$$\sqrt{x} - (\sqrt{x})_p < 1/10^p,$$

$$\sqrt{a}(a_1-a) + \sqrt{a_1}(a_2-a_1) + \ldots$$

$$+ \sqrt{a_r}(a_{r+1}-a_r) + \ldots + \sqrt{a_n}(b-a_n) - \int_a^b (\sqrt{x})_p\,dx$$

$$< \{(a_1-a) + (a_2-a_1) + \ldots + (b-a_n)\}/10^p = (b-a/10^p$$

and so $\sum_0^n \sqrt{a_r}(a_{r+1}-a_r)$ also converges to $\int_a^b \sqrt{x}\,dx$. This suggests the following definition of the definite integral for a function $f(x)$, continuous in (a, b):

Let the points $a=a_0, a_1, a_2 \ldots a_n, a_{n+1}=b$ divide the interval

(a, b) into parts sufficiently small to ensure $(f(x)-f(x'))_p=0$ for any x, x' in the same part, and let $S_p=\sum_0^n f(a_r)(a_{r+1}-a_r)$. Then $S_1, S_2, S_3, \ldots$ is convergent, and if the limit of the sequence is S we define

$$\int_a^b f(x)\,dx=S.$$

First we prove that $S_1, S_2, S_3, S_4, \ldots$ is convergent. Take any $q>p$ and, for each r, let $x_0^r=a_r$, x_1^r, $x_2^r, \ldots x_k^r$, $x_{k+1}^r=a_{r+1}$, divide the interval (a_r, a_{r+1}) into sufficiently small parts to ensure $(f(x)-f(x'))_q=0$ for any x, x' in the same part. Then

$$S_p=\Sigma f(a_r)(a_{r+1}-a_r)$$

and
$$S_q=\sum_{r,s} f(x_s^r)(x_{s+1}^r-x_s^r)=\sum_r\Big(\sum_s f(x_s^r)(x_{s+1}^r-x_s^r)\Big).$$

Since $\sum_{s=0}^{k}(x_{s+1}^r-x_s^r)=a_{r+1}-a_r$,

$$S_q-S_p=\sum_r\Big(\sum_s(f(x_s^r)-f(x_0^r)(x_{s+1}^r-x_s^r)\Big).$$

But x_s^r and x_0^r are both in the interval (a_r, a_{r+1}) so that

$$|f(x_s^r)-f(x_0^r)|<1/10^p,$$

and therefore

$$|S_q-S_p|<\sum_{r,s}(x_{s+1}^r-x_s^r)/10^p=(b-a)/10^p,$$

which proves that the sequence $S_1, S_2, S_3, \ldots$ is convergent.

Consider next any other chain of points $a=z_0, z_1, z_2, \ldots z_m, z_{m+1}=b$ dividing (a, b) into small parts in each of which $(f(x)-f(x'))_p=0$ and let $S_p'=\Sigma f(z_r)(z_{r+1}-z_r)$. We show that $S_1', S_2', S_3', \ldots$ also converges to $\int_a^b f(x)\,dx$.

Let the chain $a=w_0, w_1, w_2, \ldots w_\mu, w_{\mu+1}=b$ contain all the points $a_0, a_1, a_2, \ldots a_{n+1}$ *and* all the points $z_0, z_1, z_2, \ldots z_{m+1}$, and let $T=\Sigma f(w_r)(w_{r+1}-w_r)$. Suppose that w_i and w_j are what the points a_r and a_{r+1} become in the new enumeration. Then

$$S_p=\Sigma f(a_r)(a_{r+1}-a_r)=\sum_{r=0}^{n}\Big(\sum_{s=i}^{j} f(w_i)(w_{s+1}-w_s)\Big),$$

and so
$$S_p-T=\sum_{r=0}^{n}\Big(\sum_{s=i}^{j}(f(w_i)-f(w_s))(w_{s+1}-w_s)\Big).$$

But w_i and w_s, for $s\leqslant j$, are both points of the interval (a_r, a_{r+1}), so that

$$|f(w_i)-f(w_s)|<1/10^p,$$

and therefore $|S_p-T|<\Sigma(w_{s+1}-w_s)/10^p=(b-a)/10^p$.

Similarly $|S_p'-T|<(b-a)/10^p$

and therefore $|S_p'-S_p|<2(b-a)/10^p$.

But $$\left|\int_a^b f(x)\,dx - S_p\right| \leqslant (b-a)/10^p,$$

so that $$\left|\int_a^b f(x)\,dx - S'_p\right| < 3(b-a)/10^p,$$

and therefore $S'_1, S'_2, S'_3, \ldots$ converges to $\int_a^b f(x)\,dx$.

Theorem 1. If $f(x)$ is continuous in (a, b), and if c lies in (a, b), then

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

Choose a subdivision $a_0 = a, a_1, a_2, \ldots a_{m+1} = c, a_{m+2}, \ldots a_n, a_{n+1} = b$, such that $(f(x) - f(x'))_p = 0$ for any x, x' in a_r, a_{r+1}, and for any r.

Then

$$\left|\int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \left(\sum_0^m f(a_r)(a_{r+1}-a_r) + \sum_{m+1}^n f(a_r)(a_{r+1}-a_r)\right)\right| \leqslant 2(b-a)/10^p$$

and $$\left|\int_a^b f(x)\,dx - \sum_0^n f(a_r)(a_{r+1}-a_r)\right| \leqslant (b-a)/10^p,$$

so that $$\left|\int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \int_a^b f(x)\,dx\right| \leqslant 3(b-a)/10^p,$$

and as this is true for any value of p the result follows.

Theorem 2. If $f(x)$ and $g(x)$ are continuous in (a, b), then

$$\int_a^b \{f(x) + g(x)\}\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

$f(x) + g(x)$ is continuous, so we may choose a subdivision

$$a = a_0, a_1, a_2, \ldots a_{n+1} = b$$

such that

$(f(x) - f(x'))_p = 0$, $(g(x) - g(x'))_p = 0$, $(f(x) + g(x) - (f(x') + g(x')))_p = 0$

for any x, x' in each (a_r, a_{r+1}).

Therefore

$$\left|\int_a^b \{f(x) + g(x)\}dx - \Sigma(f(a_r) + g(a_r))(a_{r+1} - a_r)\right| \leqslant (b-a)/10^p$$

$$\left|\int_a^b f(x)\,dx - \Sigma f(a_r)(a_{r+1} - a_r)\right| \leqslant (b-a)/10^p$$

$$\left|\int_a^b g(x)\,dx - \Sigma g(a_r)(a_{r+1} - a_r)\right| \leqslant (b-a)/10^p,$$

whence

$$\left|\int_a^b (f(x)+g(x))\,dx - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right| \leqslant 3(b-a)/10^p$$

and the result follows.

Theorem 3. If $f(x)$ is continuous in (a, b) and if $m<f(x)<M$, then

$$m<\frac{1}{a-b}\int_a^b f(x)\,dx<M.$$

For if $\phi(x)$ is positive then $\Sigma\phi(a_r)(a_{r+1}-a_r)$ is positive and so $\int_a^b \phi(x)\,dx$ is positive. Take $\phi(x)=f(x)-m$ and $\phi(x)=M-f(x)$ in turn and the result follows.

Theorem 4. $\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx.$

If $f(x)$ is continuous in (a, b), since

$$|f(x)-f(x')| \geqslant ||(f(x)|-|f(x')||,$$

$|f(x)|$ is continuous. Moreover

$$|\Sigma f(a_r)(a_{r+1}-a_r)| \leqslant \Sigma|f(a_r)|(a_{r+1}-a_r).$$

Theorem 5. If $f(x)$ is continuous in (a, b) then $\int_a^t f(x)\,dx$ is differentiable at any point t in (a, b) and the derivative is $f(t)$.

Denote $\int_a^t f(x)\,dx$ by $F(t)$.

Choose a point t_0 in (a, b) and let t be another point in (a, b), such that $t-t_0$ is small enough to ensure $|f(x)-f(t_0)|<1/10^n$ for any x between t_0 and t. (We may suppose k_n chosen so that $(t-t_0)_{k_n}=0$.) Since $f(x)-f(t_0)$ is continuous,

$$\left|\frac{1}{t-t_0}\int_{t_0}^t \{f(x)-f(t_0)\}\,dx\right| \leqslant \frac{1}{|t-t_0|}\int_{t_0}^t |f(x)-f(t_0)|\,dx<1/10^n.$$

But $$\int_{t_0}^t \{f(x)-f(t_0)\}\,dx=\int_{t_0}^t f(x)\,dx-(t-t_0)f(t_0),$$

and therefore $$\left|\frac{1}{t-t_0}\int_{t_0}^t f(x)\,dx-f(t_0)\right|<1/10^n.$$

Now $$\frac{F(t)-F(t_0)}{t-t_0}=\frac{1}{t-t_0}\left(\int_a^t f(x)\,dx-\int_a^{t_0} f(x)\,dx\right)$$

$$=\frac{1}{t-t_0}\int_{t_0}^t f(x)\,dx,$$

so that $$\left\{\frac{F(t)-F(t_0)}{t-t_0}-f(t_0)\right\}_n=0, \text{ provided } (t-t_0)_{k_n}=0,$$

and as this is true for any n, it proves that $f(t_0)$ is the derivative of $F(t)$ at the point t_0.

If $G(t)$ is any function whose derivative is $f(t)$ (so that $G(t)$ is continuous), then since $G(t)$ and $F(t)$ have the same derivative, $G(t)=F(t)+C$, i.e. $G(t)-C=\int_a^t f(x)dx$. We can choose t so that $|G(a)-G(t)|<1/10^{n+1}$ and $\left|\int_a^t f(x)dx\right|<1/10^{n+1}$ and therefore, for any n, $|G(a)-C|<1/10^n$ so that $C=G(a)$. This completes the proof that if $G(t)$ is any function whose derivative is $f(t)$, then $\int_a^t f(x)\,dx=G(t)-G(a)$ and so we have established the familiar connection between the definite and indefinite integral.

The theorem that two functions which have the same derivative differ only by a constant is an immediate consequence of the Mean Value Theorem and the Mean Value Theorem depends only upon Rolle's theorem. If the definition of *differentiability* is that $F(x)$ is said to be differentiable in (a, b) if $F(x)$ is continuous in (a, b), and if we can determine a function $f(x)$ such that for any n and for any x, $x+h$ in (a, b)

$$\left(\frac{F(x+h)-F(x)}{h}-f(x)\right)_n=0 \text{ provided } (h)_{k_n}=0,$$

for a certain k_n depending only upon n (and a, b), then it is readily proved that the derived function $f(x)$ is continuous and therefore Rolle's theorem is merely an instance of the Continuity Theorem 2 we proved above.

The treatment of the fundamental theorems which we have indicated is fully as rigorous as the Dedekind-Russell formulation of the theory of functions, and yet essentially more elementary. The difficult step for the student is the determination of the limit of a convergent sequence when the calculation is complicated by the appearance of recurring 9's, but once this has been apprehended the difficulties which remain lie, not in the ideas, but in the necessity for a notation of sufficient multiplicity; it is here that the teacher's skill in making the least possible use of prefix and suffix can play an important part.

The defect of the treatment, from my own view-point, is that, though more constructive than the classical treatment (we show how to *find* the point where the continuous function vanishes, not just that its non-existence leads to a contradiction, etc.), it none the less makes frequent appeal to undecidable disjunctions. For instance in establishing the limit of a convergent sequence we spoke of a certain three possibilities and there may be cases when we cannot decide which of these possibilities in fact occurs. The determination of the point at which a continuous function takes the value zero required us to be able to say of a function $f(x)$ whether, at some point X, $f(x)$ is greater than zero or not greater than zero, and this we may be unable to decide, unless $f(x)$ is a rational function. Of course the Dedekind treatment also makes

use of undecidable disjunctions so the defect is not peculiar to this treatment alone. I am, however, convinced that it is not possible to avoid the use of undecidable disjunctions without a complete sacrifice of the conventional outward form of mathematical analysis. It may eventually prove expedient to take this step, but for the present the sole justification for a change in teaching practice must lie in the gain in simplicity for the student.

It might be profitable even for students reading Special Honours in Mathematics to be introduced to Function-theory in this way, though in writing this paper it is the needs of the ordinary, three-subject, degree student that I have in mind. I do not believe that such students should be *examined* in the fundamental theorems of analysis but I do believe that degree students ought to be sufficiently intelligent and mature in their final year to master the technique of the calculus the better for having seen something of the ideas underlying that technique. It seems to me most undesirable to award a University degree in mathematics to that level of intelligence which acquires the technical tricks of a subject the more readily for lack of the inclination to doubt and the talent for reflection.

R. L. GOODSTEIN.

CORRESPONDENCE.

SIR,—On p. 18 of the February *Gazette* is a quotation from *Pure Mathematics*. I hope that readers will have appreciated that this was intended not as a separate Gleaning—it would be an impertinence to present a sentence from this source as if it could be unfamiliar—but solely for comparison with the quotation which precedes it. Whatever the cause of the slovenly teaching in elementary analysis which was all but universal in England until Professor Hardy directed his expository genius to its eradication, it was not that his standard of accuracy in thought was unprecedented here. In Gleaning 1316 (Vol. XXIV, p. 181) was evidence that in 1842 Cambridge was, so to speak, already waiting for Weierstrass.

Yours truly,

E. H. NEVILLE.

SIR,—Is there any name, approved by the Mathematical Association, for a non-rectangular parallelogram, *i.e.* a figure which may be either rhombus or rhomboid? If not, can anybody suggest one suitable for pupils aged about 10 years?

Yours truly,

R. S. WILLIAMSON.

OCTONARIA

By Peter Simple

"*And if I have done well, and as is fitting the story, it is that which I desired; but if slenderly and meanly, it is that which I could attain unto.*"—Apocrypha

It was quite clear that the company were much more interested in what the Baron had to say about the marriage customs of the people of Octonaria than in his remarks about their system of numbering. So I bided my time until he left the hospitable parlour of *The Dolphin* and, attaching myself to him, with little difficulty persuaded him to accompany me to my lodgings. Here, comfortably ensconced in my own armchair, he proceeded to fill the great octagonal pipe which had excited such curiosity at *The Dolphin*, while I set about collecting the ingredients for a bowl of punch.

I wish I could set down in full the wonderful tale the Baron told me. The detailed information which he had collected in his travels was truly astonishing, and what a genius he had for making the most trivial occurrences seem interesting! I have endeavoured to commit to writing the gist of what he told me, but how much better this account would have been if it could have been given in the Baron's own words, and enlivened with his anecdotes!

It seems that the peculiar system of numbering adopted by the Octonarians arose quite naturally, because Octo I, the grandfather of the present king, was born with only three fingers on each hand instead of the usual four. He could not count up to ten upon his fingers and, when he came to the throne, he decided that no one else should—in fact he made it quite clear that any nonconformity would be rectified by the removal of some superfluous fingers.

To simplify the Baron's description of the system, I shall for the present omit the various names and symbols used by the Octonarians. Their original system was, indeed, very similar to our own, and when Octo the Wise made his great simplification he just abolished the number nine (except for medical purposes) and made *10* represent eight. How quickly and easily the change was effected and how readily the existing weights and measures were adapted so as to fit in with the new system!

What first excited the Baron's interest seems to have been the ease with which the little Octonarians learnt their multiplication table. He knew that they were not particularly bright or quick-witted, and when he enquired further he found that the multiplication table really was very short and simple. Here it is (Table 1), and there are only twenty-one products to be learnt.

You would have thought that Octo would have had to invent a lot of new names, one for *20*, or twice eight, for example, but he was not bothered. "We will just call it twenty", he said (in Octonarian, of course), "for we sha'n't want to call *24* twenty and it's a pity to waste a good name." *1000* (*i.e.* $8 \times 8 \times 8$), he called ten hundred;

1	2	3	4	5	6	7	1
	4	6	10	12	14	16	2
		11	14	17	22	25	3
			20	24	30	34	4
				31	36	43	5
					44	52	6
						61	7

TABLE 1
Octonarian Pinax

but a hundred hundreds, *10000*, (*i.e.* $8 \times 8 \times 8 \times 8$), he termed a myriad. A myriad myriads also received a name, but I forget what it was.

The Octonarial Association for the Advancement of Science naturally helped King Octo a good deal in preparing his subjects for the change. It was all very discreetly done, the inherent naturalness of the ordinary weights and measures being emphasised, such as :

8 pints = 1 gallon,
8 gallons = 1 bushel,
8 bushels = 1 quarter,

a naturalness which had flourished in spite of denary arithmetic.

1	2	3	4	5	6	7	8	9	↊	↋	1
	4	6	8	↊	10	12	14	16	18	1↋	2
		9	10	13	16	19	20	23	26	29	3
			14	18	20	24	28	30	34	38	4
				21	26	2↋	34	39	42	47	5
					30	36	40	46	50	56	6
						41	48	53	5↊	65	7
							54	60	68	74	8
								69	76	83	9
									84	92	↊
										↊1	↋

TABLE 2
Duodecimal Pinax

The coinage seems to have helped a little : being relatively poor, the Octonarians were chiefly interested in petty cash and they had found it convenient to issue, as we do for Gibraltar, half-farthings and quarter-farthings. There was also a silver *groat* (4d.).

Curiously enough, counter-propagandism was almost entirely confined to the activities of an insignificant group which, because of the supposed advantages of division into three, favoured a duodecimal system. The teachers, however, opposed this system as soon as they saw the proposed multiplication table (Table 2). The Octonarians, moreover, had very little use for division into three —particularly when they remembered the penalty for nonconformity.

I must confess that I did not realise, until the Baron pointed it out to me, that weights and measures ought always to be based upon the scale of two, and that the *metric* system is really a most unscientific affair. The seven metric weights, 1, 2, 2, 5, 10, 10 and 20 grammes, will weigh up to only 50 grammes, but a similar number of avoirdupois weights, 1, 2, 4, 8 ounces, 1, 2 and 4 pounds, weigh up to 127 ounces.

Changes in the measures of length seem to have worried the Octonarian engineers far less than a change to the metric system would have done. Not only did they retain their inches, their eighths and their beloved sixty-fourths, but they found to their delight that they were granted also what they had always deemed to be the exclusive privilege of the metric system, the expression of fractions by means of *Napier's point* ; one inch and three-eighths became 1·3 in., and one inch and seven sixty-fourths 1·07 in.

The Baron showed me an Octonarian slide-rule, and it seemed to me that the graduations ran more smoothly than those of a common slide-rule. He had, too, an Octonarian watch, the hours of which were sixty-fourths of a day, each hour being divided, when necessary, into eighths and sixty-fourths.

I enquired further about the coinage and learnt that the Octonarian *crown* (5s.) used to be divided into sixty pence. This crown has been retained and the present penny is 15/16ths of the old one. There is a gold coin equivalent to 8 crowns (£2) and, according to the Baron, higher multiples of this are known by the names of animals : I remember that he talked of *ponies* and *monkeys*, but forget what each denoted. This may be an indication of Greek influence, for the Greeks stamped some of their coins with owls and tortoises.

That the people of Octonaria are well satisfied with their present regime seems to be evidenced by the pious expression with which they now end their proclamations :

GOD PRESERVE
THE KING'S
DIGITS

A STAGGERING PROPOSITION.

By N. M. Gibbins.

[This article is written by a master of a boys' school, but the sexes of the teachers and the children may be varied by the reader in any way desired.]

The question of the length of masters' holidays in a reception area is a somewhat thorny one, the essential factor being that the boys must be looked after the whole year round. In order to fix our ideas it is convenient to consider two extreme cases. First, let us suppose it is decided that the masters are to have the same length of holiday period as the boys. If a programme of work can be arranged with two-thirds of the staff on duty, and if the boys can be properly supervised when on holiday by one-third of the staff, then diagram I shows a possible solution for a holiday of four weeks. In the diagram the vertical lines separate a Saturday and Sunday, and the holiday periods begin on a Wednesday and end on a Thursday ; also the letters refer to approximately equal groups of masters. It is clear that this arrangement throws a very great strain on the duty-staff, and this may possibly offset to some extent the benefit of the holiday. In any case the normal work of the school is interrupted for eight weeks instead of four.

Secondly, let us assume that normal work with full staff goes on right up to the end of a term and from the beginning of the next. It has to be decided whether half the staff, or a smaller proportion —say one-third as before—is enough for the purpose of looking after the boys during their holiday. If one-half, then only one-half of the boys' holiday period can be had by the masters ; and this is too short for recuperation. If one-third, then two-thirds of the boys' holiday period can be had. But this necessitates one group of masters splitting their share into two halves as in diagram II, which for convenience shows a boys' holiday period of three weeks. But it is of the essence of a satisfactory holiday that it should be continuous, and not to be broken into except for a grave emergency, and in that event all masters will be recalled to deal with it. Besides double travelling expenses may have to be incurred by those who split their holiday.

The major weakness of either of these plans is that, with the majority of the boys being kept in the reception area, it is highly improbable that one-third of a staff is sufficient for holiday supervision ; so that, in what follows, one-half of the staff will be assumed to be on duty during the holidays, and three-quarters for the inevitable periods of dislocation before and after ; while an exception to this rule may perhaps be allowed for certain week-ends.

The essence of the compromise to be considered is that there should be the minimum upset, as regards its length and severity, before and after the boys' holiday, and the maximum length of holiday for the masters.

	aaaa	*aaaaaaa*	*aaaaaaa* *bbbb*	*aaaaaaa* *bbbbbbb*	*aaa* *bbbbbbb* *ccc*	*bbbbbbb* *ccccccc*	*bbb* *ccccccc*	*ccccccc*	*ccc*
I.			↓	Boys' holiday: 4 weeks		↑			
			aaaa *bbbb*	*aaa* *bbbbbbb*	*aaaa* *bbb* *ccc*	*aaa* *ccc*			
II.			↓	Boys' holiday: 3 weeks		↑			
	aaaa	*aaaaaaa* *bbbb*	*aaa* *bbbbbbb* *ccc*	*bbb* *ccccccc* *ddd*	*ccc* *ddddddd*	*ddd*			
III.			↓	Boys' holiday: 3 weeks		↑			
	aa	*aaaaaaa* *b*	*aaaaaaa* *bbbbbbb* *ccc*	*aaa* *bbbbbbb* *ddd*	*bbbb* *ccccccc* *ddddddd*	*ccccccc* *ddddddd*	*c* *ddddddd*	*dd*	
IV.			↓	Boys' holiday: 30 days		↑			
	a	*aaaaaaa* *b*	*aaaaaaa* *bbbbbbb* *c*	*a* *bbbbbbb* *ccccccc*	*b* *ccccccc* *ddddddd*	*ccccccc* *ddddddd*	*c* *ddddddd*	*d*	
V.			↓	Boys' holiday: 23 days		↑			
	a	*aaaaaaa* *b* *c*	*a* *bbbbbbb* *ccccccc* *d*	*b* *ddddddd*					
VI.			↓	Boys' holiday: 9 days		↑			

The period of upset or dislocation is assumed to be about a week, and during this time it is found in practice that the boys need not lose much of their teaching. Taking a nine-period day as normal, with a period of 35 minutes, and omitting the last period for this time of strain, it often happens that the last lesson of a man who is present can be fitted into the period vacated by an absentee, while at the other end of the rota a similar thing happens, *mutatis mutandis*. The proportion of time devoted to the various subjects is roughly maintained, though the total number of periods is slightly reduced.

Diagrams III and IV show plans for boys' holidays of three and four weeks respectively. In the latter, advantage is taken of week-ends to add two days to the boys' month, and this necessitates an extra day's upset at each end for the masters to have nineteen days' holiday on end.

If it be assumed that a quarter of the staff is enough to look after the boys on Saturday and Sunday when the duties are usually fewer, a considerable advantage can be taken of week-ends in the masters' interests, while the boys gain two days. Diagram V shows the possibility for a holiday of three weeks : the boys getting 23 days and the masters 16 days, while the two middle week-ends are short-staffed. The same principle holds for a five weeks boys' holiday, giving them 37 days and the masters 23 days ; or for a seven weeks boys' holiday, giving them 51 days and the masters 30 days. At the other end of the scale we can have a most unsatisfactory " nine days' wonder " for all, as shown in diagram VI.

It is admitted that the short-staffed week-ends are, in general, a weakness in this compromise ; but in some schools masters have established homes in the reception area, and if it can be arranged that a number of those on holiday are in residence, they will be available in a minor emergency—for boys persist in having accidents at the most inconvenient times. Moreover, these masters will be available for a major emergency while the rest of the staff are being recalled.

To sum up, it is clear that masters cannot look after the boys the whole year round and at the same time have the periods of recuperation of normal times. In the long run this is bound to result in loss of efficiency, and is the example, in the teaching profession, of the general truth that war results in waste.

N. M. G.

1372. In effect it was an isosceles triangle which was being attacked along the whole length of its long sides. The short base was Ypres itself. . . . The main road, along which all the traffic of reinforcement and supply went, bisected the base of the triangle and ended at its point. Some cynic remarked, truly enough, that if you walked from Ypres along this road you could be hit from almost every possible angle except from directly behind, so that the only invulnerable part of your body would be a long thin line down the spine.—*Vain Glory*, p. 141, a miscellany of the Great War, edited by Guy Chapman. [Per Mr. P. J. Harris.]

MATHEMATICAL NOTES.

1532. *The orthogonal projection of a triangle into an equilateral triangle.*

Suppose we project a triangle ABC, with sides a, b, c into an equilateral triangle of side d, the projection being orthogonal. If the heights of the vertices of ABC above the plane of projection are respectively y, $y+y_1$, $y+y_2$, application of Pythagoras' theorem gives us the three equations

$$a^2=d^2+(y_1-y_2)^2, \quad b^2=d^2+y_2^2, \quad c^2=d^2+y_1^2,$$

Elimination of y_1, y_2 then gives the following equation for the side of the equilateral triangle

$$3d^4-2d^2(a^2+b^2+c^2)+16\Delta^2=0.$$

This result is, of course, well known and has been set in examinations. Further examination suggests a number of additional questions! We know that a conic can be drawn through five points. We draw one touching the sides AB, BC respectively of the triangle ABC at their midpoints, and passing through the midpoint of CA. A variety of theorems suggest that this conic will also touch CA at its midpoint. Now, if this conic were an ellipse, we could project it orthogonally into a circle, and ABC would project into a triangle whose sides are bisected at the points of contact of the incircle. The triangle is therefore equilateral.

Is the unique conic which touches the sides of ABC at their midpoints always an ellipse? If not, we cannot always project ABC into an equilateral triangle. We look at our equation for d. The roots are both positive, if real, and the condition for reality is $a^2+b^2+c^2 \geqslant 4\sqrt{3}\Delta$. Is this condition always satisfied? We go back to our conic, and convince ourselves that it is always an ellipse, if needs be by obtaining its areal equation. It follows that for any triangle $a^2+b^2+c^2 \geqslant 4\sqrt{3}\Delta$.

Examining this condition more closely, we find that it is equivalent to

$$a^4+b^4+c^4 \geqslant b^2c^2+c^2a^2+a^2b^2,$$

i.e. to

$$(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2 \geqslant 0,$$

so that the equality sign only holds when ABC is itself equilateral.

One further question. Why does the equation for d give *two* real positive values for d? We know there is only one! If θ is the angle between the plane ABC and the plane of projection, since the area of the equilateral triangle is $\frac{1}{4}\sqrt{3}d^2$, we have $\frac{1}{4}\sqrt{3}d^2=\Delta\cos\theta$. The equation for $\cos\theta$ is therefore

$$\cos^2\theta-\frac{(a^2+b^2+c^2)}{2\sqrt{3}\Delta}.\cos\theta+1=0.$$

The product of the roots of this equation is equal to one. If one root is less than one, the other must be greater than one. Although

this latter root does not correspond to a real angle of projection, it corresponds to a formal orthogonal projection in which the minor axis of the ellipse inscribed in *ABC increases* to the length of the radius of the incircle of the equilateral triangle. D. PEDOE.

1533. *Feuerbach's theorem.*

1. I give here a simple geometrical proof of Feuerbach's theorem ; although I have collected more than 60 geometrical proofs of this theorem, this proof has not, to my knowledge, been previously published.

In December 1903, Mr. C. E. Youngman stated and proved a theorem which was afterwards published in *Mathematical Questions and Solutions from the "Educational Times"*, New Series, Vol. VI (1904), p. 101, No. 14845 (see 3 of this Note). After giving his own solution, Mr. Youngman points out that the theorem furnishes immediate proofs of several other theorems and enumerates five of them, but Feuerbach's theorem is not one of them, though it follows almost as a corollary from a particular case of Youngman's theorem.

2. Mr. Youngman's theorem may be stated as follows : (I have altered some of the letters.)

In a triangle *ABC*, if *DEF* be the pedal triangle of the circumcentre *S*, and *KLM* the pedal triangle of any other point *J* on a fixed diameter *SJ*, then :

(i) there exists a point *O* making the triangles *ODK*, *OEL*, *OFM* similar to *ASJ*, *BSJ*, *CSJ* respectively ;

(ii) the circle *KLM* passes through *O* wherever *J* may be on *SJ*.

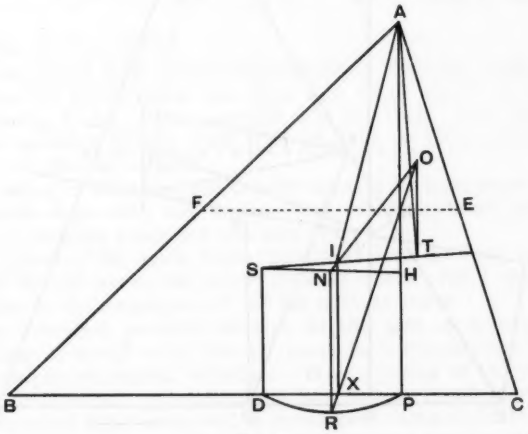


FIG. 1.

Now let us take the incentre I as a special case of the point J. By drawing AT perpendicular to SI and taking the reflection of T in EF, we get Youngman's point O for the diameter SI. Let H be the orthocentre, X the point of contact of the incircle with BC, N the centre of the nine-points circle, and P the foot of the altitude from A. Then by Youngman's theorem the incircle and the nine-points circle pass through O.

Also by the same theorem,

$$\begin{aligned}\angle DOX &= \angle SAI \\ &= \tfrac{1}{2}\angle SAH \\ &= \tfrac{1}{2}\angle DOP \text{ by Youngman's theorem.}\end{aligned}$$

Thus OX bisects the arc DP of the nine-points circle, say at R, and NR is perpendicular to BC.

Hence $$\angle IOX = \angle IXO = \angle NRO = \angle NOR.$$

Thus N, I, O are collinear, and hence

$$NI = NO - IO = \tfrac{1}{2}R - r$$

and the circles touch at O.

3. For the convenience of readers who may not possess a copy of *Educational Times*, Vol. VI, I give a simple proof of Youngman's theorem.

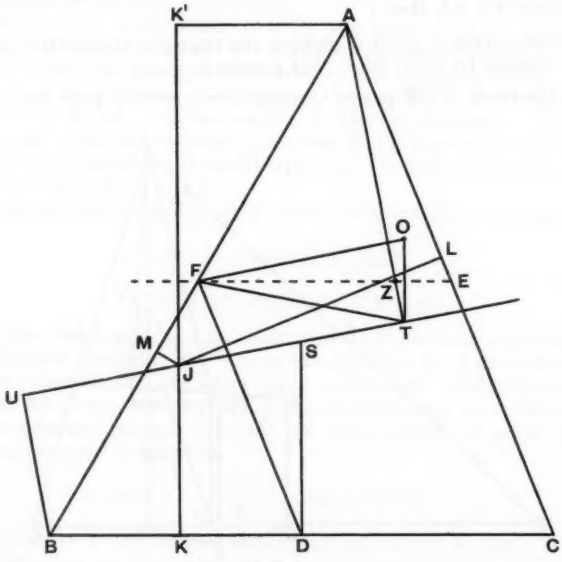


FIG. 2.

Draw AT, BU perpendicular to SJ and take O the reflection of T in EF. Let K' be the reflection of K in EF. Let AT cut EF at Z.

Then $$FO = FT = FU.$$

Also
$$\begin{aligned}\angle DFO &= \angle DFE + \angle EFO \\ &= \angle DFE + \angle EFT \\ &= \angle AEF + \angle TAE \\ &= \angle AZF \\ &= \angle DSU \text{ (since } DS \text{ is perpendicular to } EF \text{ and } SJ \text{ to } AT) \\ &= \angle DFU.\end{aligned}$$

Thus O is the reflection of U in DF.

(i) $\angle OKD = \angle TK'A = \angle TJA = \angle AJS.$

Similarly $\angle ODK = \angle ASJ$ and the triangles OKD, AJS are similar. Similarly OLE, BJS and OMF, CJS are similar.

(ii) Again,
$$\begin{aligned}\angle KOL + \angle C &= \angle OKD + \angle OLE \\ &= \angle AJS + \angle BJS, \text{ by (i)} \\ &= \angle AJB \\ &= \angle AJM + \angle BJM \\ &= \angle ALM + \angle BKM \\ &= \angle KML + \angle C.\end{aligned}$$

Thus $\angle KOL = \angle KML$, and O is on the circle KLM.
Note that O is the orthopole of the diameter SJ.

SIR V. RAMESAM.

1534. *On Note* 1509.

With regard to Mr. Styler's entertaining Note No. 1509, it may perhaps be pointed out that over any single period of 23 years containing 5 leap years, payment will be exact; since the length of the period is 23×52 weeks plus 28 days, or 1200 weeks: while $1200 \times £1\ 18s.\ 4d. = £2300$.

A period of 46 years will usually contain 11 leap years, making 57 extra days—one too many. But exact payment will result when it includes a century non-leap year.

A period of 69 years must contain at least 16 leap years and hence 69×52 weeks + 85 days, or 3600 weeks plus 1 day. Thus the gain to the employee is 1/252 per cent. at worst.

The strongest possible attempt on the part of a firm of long standing to avoid such over-payment to a succession of equally paid employees results in failure. Over a period of 115 years, e.g. from 1797 to 1911 (which contains two century non-leap years, and during which inflation as well as investment value are to be ignored) there must be at least 26 leap years, making a total of 6000 weeks

plus 1 day. Here the collective profit of the employees is reduced to 1/420 per cent. N. M. G.

1535. *The Converse of Apollonius' Theorem.*

The use of the converse of Apollonius' Theorem by a pupil attempting a rider led me to investigate its truth, and provided the material for an interesting lesson.

Given. X is a point on the base BC of a triangle ABC such that $AB^2+AC^2=2AX^2+2BX^2$.

Is AX necessarily a median?

Construction. Join X to O, the mid-point of AB (Fig. 1).

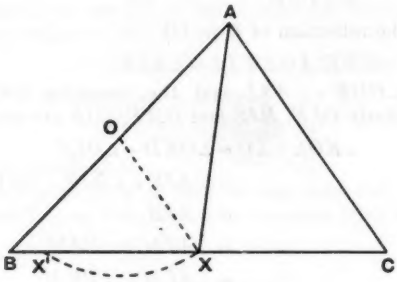


FIG. 1.

Then $$AX^2+XB^2=2OX^2+2OA^2,$$

whence $$AB^2+AC^2=4OX^2+4OA^2.$$

But $AB^2=4OA^2$, so that $OX=\frac{1}{2}AC$.

There are, therefore, in general two points on BC satisfying the condition; these are the points in which the circle, centre O, radius $\frac{1}{2}AC$, cuts BC.

Investigation of the disappearance of the ambiguity proved exceedingly interesting.

1. It is clear that if $OX>OB$, i.e. $AC>AB$, X' lies on CB produced, which is contrary to hypothesis.

2. If $C=90°$, the circle touches BC, and AX *must* be a median.

3. If $C>90°$, it may happen that X' lies on BC produced (Fig. 2). This occurs when $OX>OC$, or $OX^2>OC^2$.

Now $$2OC^2+2OA^2=AC^2+BC^2,$$

or $$OC^2=\tfrac{1}{2}a^2+\tfrac{1}{2}b^2-\tfrac{1}{4}c^2;$$

so that X' lies on BC produced if $\frac{1}{4}b^2>\frac{1}{2}a^2+\frac{1}{2}b^2-\frac{1}{4}c^2$,

i.e. if $c^2>2a^2+b^2$ .....(i)

But $c^2 = a^2 + b^2 + 2a \,.\, CD$, so that (i) is equivalent to

$$a^2 + b^2 + 2a \,.\, CD > 2a^2 + b^2,$$

or
$$CD > \tfrac{1}{2}a \qquad \text{.....(ii)}$$

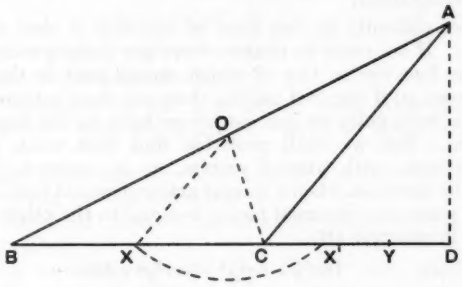


FIG. 2.

Trigonometrically, (ii) is equivalent to

$$b \cos (A + B) > \tfrac{1}{2}a,$$

or
$$2 \sin B \cos (A + B) > \sin A,$$

i.e.
$$\sin (A + 2B) - \sin A > \sin A,$$

or
$$\sin (A + 2B) > 2 \sin A. \qquad \text{.....(iii)}$$

From (iii) it is clear that, (C being obtuse), there must be ambiguity if $A > 30°$.

The results of §§ 1, 2, 3 (ii) show that, BC being considered fixed, ambiguity occurs if A lies anywhere between the perpendiculars to BC through its mid-point, and through the point Y on BC produced such that $CY = \tfrac{1}{2}BC$, except on the perpendicular through C.

A. J. G. MAY.

1536. *Some Notes on Factorisation.*

One consequence of the amount of time devoted to Mathematics in Secondary schools is that, even in schools with a single-form entry, one master can hardly cover all the work and there may be surprises awaiting some of us if we start preparing a lesson on some quite elementary work which it has not previously fallen to us to teach. This was my experience when, after seventeen years' teaching, a redistribution of forms gave me for the first time the task of introducing a form to factorisation. I had of course an adequate practical facility in recognising the different types of expression likely to be set for factorisation in examinations and spotting their factors. When, however, the necessity of teaching the methods impelled me to analyse them I was surprised to find how haphazard they were. Further analysis led to the following

three points of systematic approach to different types of factor questions which I have never seen stressed or even mentioned in any textbook.

I. *Expressions of four terms factorised as the product of two binomials by pairing terms.*

The chief difficulty in this type of equation is that of grouping the terms. If we trust to chance there are three possible ways of pairing the four terms, two of which should lead to the discovery of the factors, so if our first pairing does not show a binomial factor common to both pairs we can simply go back to the beginning and start again. But we shall probably find that weak pupils will either continue, with absurd results, as in example (A) below, abandon the question when a second attempt would have succeeded, or restart when one binomial factor is equal to the other multiplied by -1 as in example (B).

(A) $18x^3y + 4a^2y^2 + 3ax^2y^2 + 24ax$

$\equiv 2y(9x^3 + 2a^2y) + 3ax(xy^2 + 8)$

$\equiv (2y + 3ax)(9x^3 + 2a^2y)(xy^2 + 8).$

(B) $ax^2 - axy + by - bx$

$\equiv ax(x - y) + b(y - x)$

$\equiv$?.

If, however, we systematise our method of arranging the terms, the second difficulty need never arise and the first rarely. The arrangement should obey three rules.

(i) The first term of the given expression should remain first.

(ii) The second term should be chosen to have as much as possible in common with the first, e.g. in expression (A) above the factor common to $18x^3y$ and $3ax^2y^2$ involves a numerical factor and two letters while that common to $18x^3y$ and $4a^2y^2$ involves only a numerical factor and one letter.*

(iii) To decide which of the other two terms should be placed third and which fourth, whatever letter occurs to a higher power in the first term than in the second should also occur to a higher power in the third than in the fourth.

e.g. in expression (B),

(i) ax^2 remains first ;

(ii) $-axy$ remains second, giving a second degree factor for the first pair ;

(iii) The first term has a higher power of x than the second so $-bx$ should come third and $+by$ fourth. The whole example is

* If the first term is unity none of the remaining terms has a factor in common with the first. The object of the grouping must then be so to choose the second term that there may be as much as possible common to the third and fourth terms but this exceptional case need not be introduced until reasonable facility in more normal cases has been acquired.

then written :

$$\begin{aligned} & ax^2 - axy + by - bx \\ \equiv & ax^2 - axy - bx + by \\ \equiv & ax(x-y) - b(x-y) \\ \equiv & (x-y)(ax-b). \end{aligned}$$

II. *Trinomial with first coefficient unity.*

In this type of question we teach :

that the coefficient of the first degree term gives the sum or difference of two numbers which we have to write in our two brackets ;

that the sign of the absolute term tells us whether the sum or the difference is given by this coefficient ;

that the numerical value of the absolute term gives the product of the two numbers.

Faced with the expression $x^2 - 3x - 10$ the class quickly spots 2 and 5 as the required numbers whose difference is 3 and whose product is 10. Faced, however, with $x^2 - 11x - 840$, do we insist that they must try successive pairs of numbers differing by 11 in search of a pair whose product is 840 and *not* various pairs of factors of 840 in search of a pair whose difference is 11? The disadvantages of the latter method are :

(i) each failure gives no guidance for the choice of the next pair of numbers to try ;

(ii) when we come to apply our knowledge to the solution of quadratic equations it does not lead us to a quick decision as to whether a given trinomial does or does not factorise.

III. *Trinomial with non-unit coefficient.*

Here the average textbook unashamedly advocates haphazard trial of the different permutations of the possible factors of the first and third terms until the required middle term is achieved. When we realise that $40x^2 + ax - 63$ factorises for 24 different numerical values of a (each with either sign) this means that, with a given coefficient of x :

(i) we have 24 different permutations to check ;

(ii) if none of them gives the correct middle term there is always the chance that a mistake has been made and we feel no certainty that the expression does not factorise until all 24 have been re-checked.

Any systematic method of factorising expressions of this type must depend on the connection between the factors of $ax^2 + bx + c$ and those of $x^2 + bx + ac$.

$$\text{e.g. } 40x^2 + 34x - 63 \equiv \frac{(40x)^2 + 34(40x) - 40 \times 63}{40}$$

$$\equiv \frac{y^2 + 34y - 2520}{40} \quad (\text{where } y = 40x)$$

$$\equiv \frac{(y+70)(y-36)}{40}$$

$$\equiv \frac{(40x+70)(40x-36)}{40}$$

$$\equiv \frac{10(4x+7)\cdot 4(10x-9)}{40}$$

$$\equiv (4x+7)(10x-9).$$

This arrangement, however, is rather cumbersome and not easy for the beginner to grasp. It remains to find some simpler way of arranging the work which still makes it dependent on the systematic checking of pairs of numbers whose difference is 34 until a pair is found whose product is 40×63. The key to the method lies in the idea that factorisation is the reverse of a multiplication which originally gave two separate first degree terms and that the numbers (70 and 36) which the check gives us are the respective coefficients of these terms. When we multiply the two binomials $4x+7$ and $10x-9$ together, the last stage is to combine the like terms $70x$ and $-36x$ into the single term $+34x$. Consequently, when we are given the final product and want to find the factors, the first step is to split the $34x$ into the $70x$ and $-36x$ of the fully expanded product. The subsequent factorising of $40x^2+70x-36x-63$ is of course a simple example of the " pairing terms " method with no need of rearrangement. The question is therefore set out as follows :

$$40x^2+34x-63$$

$$\equiv 40x^2+70x-36x-63$$

$$\equiv 10x(4x+7)-9(4x+7)$$

$$\equiv (4x+7)(10x-9).$$

As for the best way of leading up to this method I incline to the following :

(i) When factorising expressions of the x^2+ax+b type we think of the two numbers we find with the required sum (or difference) and product as the numbers to be placed in the brackets of the answer. Checking several such answers by multiplying out we find that they are also the coefficients of the two separate x-terms in the expansion.

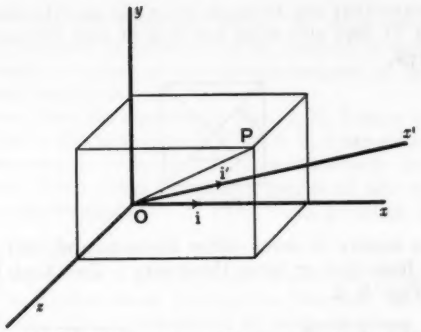
(ii) In the more general case we emphasise that although we can no longer find " in our heads " the numbers to put in the brackets we can still find the separate x-terms of the fully expanded product.

T. H. F.

1537. *Change of axes.*

If P is (x, y, z) referred to oblique axes and $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are *unit* base vectors, then

$$\overrightarrow{OP} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z.$$



Let $\mathbf{i}^*$, $\mathbf{j}^*$, $\mathbf{k}^*$ be the reciprocal system (not necessarily unit vectors), i.e.

$$\mathbf{i}^* = \frac{\mathbf{j} \wedge \mathbf{k}}{[\mathbf{i}\ \mathbf{j}\ \mathbf{k}]}, \text{ etc.},$$

where $[\mathbf{i}\ \mathbf{j}\ \mathbf{k}]$ denotes the triple scalar product of the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Then $\mathbf{i}\mathbf{i}^* = 1$, $\mathbf{j}\mathbf{i}^* = 0$, $\mathbf{k}\mathbf{i}^* = 0$. Hence $x = \mathbf{i}^* \overrightarrow{OP}$.

Thus we can change to any other system of axes ($\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$) by remarking that

$$x = \mathbf{i}^*(\mathbf{i}'x' + \mathbf{j}'y' + \mathbf{k}'z'),$$

$$\mathbf{i}^*\mathbf{i}' = \frac{[\mathbf{i}'\ \mathbf{j}\ \mathbf{k}]}{[\mathbf{i}\ \mathbf{j}\ \mathbf{k}]}, \quad \mathbf{i}^*\mathbf{j}' = \frac{[\mathbf{j}'\ \mathbf{j}\ \mathbf{k}]}{[\mathbf{i}\ \mathbf{j}\ \mathbf{k}]}, \quad \mathbf{i}^*\mathbf{k}' = \frac{[\mathbf{k}'\ \mathbf{j}\ \mathbf{k}]}{[\mathbf{i}\ \mathbf{j}\ \mathbf{k}]}.$$

L. M. MILNE-THOMSON.

1538. *Tangrams and incommensurables.*

Dudeney, in both his *Amusements in Mathematics* and his *Puzzles and Curious Problems*, mentions Tangrams. He remarks that they are supposed to be of Chinese origin (I have been told 2000 B.C.) and that no one has been able to improve upon the original. My own efforts certainly confirm the last statement; I ended with a profound admiration of the genius who made the first set.

Yet here are some extensions that may perhaps be of use in the air-raid shelter. The incident that set me off on this track will also give food for thought and enquiry.

One class was divided into two groups, those to whom the idea of incommensurables seemed obvious and those to whom it very definitely did not. The division in this case had no connection with the usual division between good and weak pupils. What led me to enquire, I cannot remember, but I found that the division was simply between those who had and those who had not met Tangrams when they were in the nursery. The connection is obvious. Have others noticed it, or was it just chance?

Improvements on the original seem impossible, but here are some extensions.

1. Find shapes that can be made from one set (the standard set is shown in Fig. 1), and also from two sets of half the size, *i.e.* of the same total area.

FIG. 1.

2. Divide a square in some other fashion, and find shapes that can be made from two or from three sets. Two that I have tried are given in Figs. 2, 3.

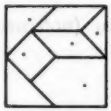


FIG. 2.

FIG. 3.

3. Find shapes that can be made from one set in several different ways.

4. Try to make hexagonal puzzles ; I tried this, in the first place, because I thought that I might get something interesting with three angles (30, 60, 90) instead of just the two of the original (45 and 90). Figs. 4, 5, 6 show three that I have tried.

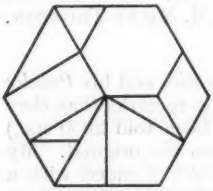


FIG. 4.

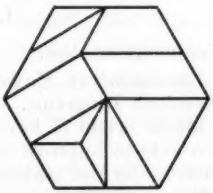


FIG. 5.

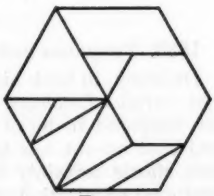


FIG. 6.

5. Find a method by which these problems can be solved by calculation ; for example, I find that Fig. 7 can be made from the

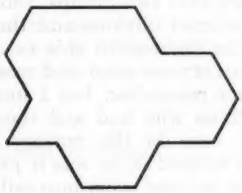


FIG. 7.

pieces of Fig. 5 in 123 different ways. Is there any means of calculating the number of possible ways, instead of proceeding by trial? The result is typical of the unexpectedness of the results of all the suggested problems.

It will be seen that the squares in Figs. 1, 2, 3 have all been cut into 7 pieces, while the hexagons in Figs. 4, 5, 6 are all in 10 pieces. Such a large number as 10 is admittedly a drawback, but is almost inevitable if one is to avoid too many pieces of the same shape, or of quite unsuitable shapes. C. DUDLEY LANGFORD.

1539. *Models of polyhedra.*

After seeing the plates of photographs of star and other polyhedra in Rouse Ball's *Mathematical Recreations* (eleventh edition) it was decided to attempt the construction of some of them. The construction of the great stellated dodecahedron is described below; and the necessary setting out for the figure of five symmetrically interlocking cubes is given (Rouse Ball, Plate II, Nos. 32, 36).

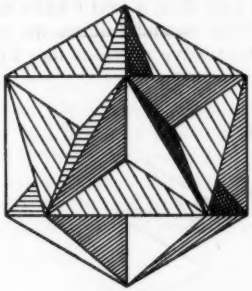


Fig. 1.

Some practical points are as follows. Thin cardboard is best, some types of exercise book covers being suitable. A fast-drying cement such as that used in model airplane construction is essential. Marking out is best done by pricking the design through from a template, and all seams should be lightly scored before cutting out. Blobs of surplus cement and ragged corners can be tidied by means of a razor blade, and a coat of enamel gives a good finish.

The great stellated dodecahedron is made from nine portions like Fig. 2 and two like Fig. 3. The first step is to join the edges *a* and *b*,

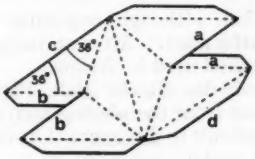


FIG. 2.

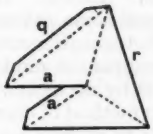


FIG. 3.

the flaps being underneath. The model is now made in three steps; the main portion is made up of a chain of five of the pieces like Fig. 2 by joining the free edge c of one to the flap d of the next and so on. The top (Fig. 4) consists of two pieces like Fig. 2 (but

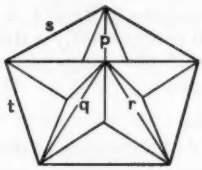


FIG. 4.

with the unlettered flaps removed) joined along the seam p and one of the pieces like Fig. 3 joined to them along q and r. It is built on to the main portion, the piece $pqts$ being added last; the edges p and s are fixed first and then q and t have to be gummed in one step. The bottom of the model follows on exactly similar lines. For the five-cube model twelve portions like Fig. 5 and twenty like

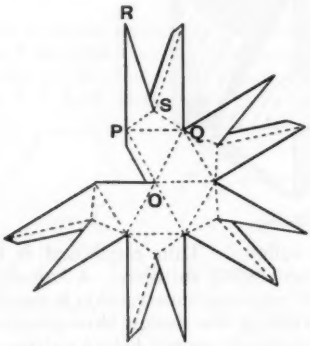


FIG. 5.

OP	59	PQ	62
OS	69	PR	100
OR	155		

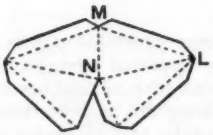


FIG. 6.

LM	100
MN	53
NL	95

Fig. 6 are needed. The construction, although requiring much joining and some patience, is straightforward. After an inspection of the photograph mentioned, the method used is obvious.

The companion model of five tetrahedra (Rouse Ball, Plate II, No. 33) is by no means easy to set out from the photograph, and a workable method of construction is difficult to discover; I have only been able to make a very indifferent model.

J. H. CADWELL.

1540. *The residue of* $(p-1)!$ *(mod* p^2*).*

1. Wilson's Theorem states that if p is prime, then $(p-1)! \equiv -1 \pmod{p}$. Hardy and Wright prove (Theorem 133 of their *Theory of Numbers*) that if $p > 2$, then

$$(p-1)! \equiv (-1)^{\frac{1}{2}(p-1)}\, 2^{2p-2} \left(\frac{p-1}{2}!\right)^2 \pmod{p^2}.$$

We can show that

$$(p-1)! \equiv -1 + p \cdot \frac{1}{2} \sum_{t=1}^{p-1} \left\{\frac{1}{p} \pmod{t}\right\} \pmod{p^2},$$

where the terms to be summed are least positive residues. We show also a general form of this result for any square modulus.

2. If m, t are coprime positive integers and a, b are the integers such that

$$(2.1) \qquad am - bt = 1, \quad 0 < a < t, \quad 0 < b < m, \quad m > 2,$$

then a, b are uniquely determined by the usual C.F. process and are the least positive residues satisfying

$$am \equiv 1 \pmod{t}, \qquad bt \equiv -1 \pmod{m},$$

we may write $a \equiv \dfrac{1}{m} \pmod{t}$.

Consider the product Πt where t takes all values less than and prime to m. Then b also takes all these values. For, (i) b is prime to m by (2.1) and (ii) different values of t must give different values of b. [$bt_1 \equiv 1 \pmod{m}$ and $bt_2 \equiv 1 \pmod{m}$ imply $t_1 \equiv t_2 \pmod{m}$ since b is prime to m]. If $m > 2$ and t is prime to m, so is $m - t$; hence the number of factors in the product Πt is even.

Now
$$\begin{aligned}(\Pi t)^2 &= \Pi t \,.\, \Pi b = \Pi bt \\ &= \Pi(am-1), \text{ where the number of factors is even,} \\ &\equiv 1 - m\Sigma a \pmod{m^2},\end{aligned}$$

that is,

$$(2.2) \qquad (\Pi t)^2 \equiv 1 - m \,.\, \sum_t \frac{1}{m} \pmod{t} \pmod{m^2}.$$

This is the general form of our result. If m is composite we cannot in general deduce that $\Pi t \equiv \pm 1 \pmod{m}$ though this is true. (Hardy and Wright, Th. 129.)

Let m be prime, $= p$ (say). Then $\Pi t = (p-1)!$ and we have

$$\{(p-1)!\}^2 \equiv 1 - p \sum_{t=1}^{p-1} \frac{1}{p} \pmod{t} \pmod{p^2};$$

hence
$$(p-1)! \equiv \pm\left\{1 - p \cdot \frac{1}{2} \sum_t \frac{1}{p} \pmod{t}\right\} \pmod{p^2}$$

(see Hardy and Wright, p. 97, § 3, on the quadratic residues of p^α). By Wilson's Theorem the minus sign must be selected. Hence

$$(2.3) \qquad (p-1)! \equiv -1 + p \cdot \frac{1}{2} \sum_t \frac{1}{p} \pmod{t} \pmod{p^2}.$$

3. For relatively small values of m the evaluation of Σa is not difficult and in practice the continued fraction algorithm can be avoided. Since every b is a t, each value of a belongs to two values of t, except when $t^2 \equiv -1 \pmod{m}$. Moreover we can easily deduce the values of a belonging to $m-t$ and $m-b$. *E.g.*

$$4 \cdot 29 = 3 \cdot 39 - 1 ;$$

hence
$$(39-4)(39-29) = (39-4-29)\,39 - 1,$$

showing that if $m = 39$, a takes the values 3, 3, 6, 6 when t takes the values 4, 29, 35, 10.

These and other simple numerical considerations are useful for a single value of m. We can, however, construct a table of values of a for values of m, t increasing from 2 by a process of repetitions, subtractions and additions in which the relative primeness of m and t is automatic.

(i) If $a_1 m \equiv 1 \pmod{t}$ and $a_2(m - Kt) \equiv 1 \pmod{t}$, then $a_1 \equiv a_2 \pmod{t}$, and if both are least positive residues, $a_1 = a_2$. That is,

$$\frac{1}{m} \pmod{t} = \frac{1}{m - Kt} \pmod{t}. \tag{3.1}$$

(ii) Remove the restriction $t < m$ and suppose $a_1 m \equiv 1 \pmod{t}$ and $a_2(t-m) \equiv 1 \pmod{t}$.

Then $a_1 \equiv -a_2 \pmod{t}$, and if both are least positive residues, $a_1 + a_2 = t$. That is,

$$\frac{1}{m} \pmod{t} + \frac{1}{t-m} \pmod{t} = t. \tag{3.2}$$

(iii) Still suppose t greater than m. Then there is a K such that

$$Km < t < (K+1)\,m.$$

Suppose
$$a_1 m \equiv 1 \pmod{t},$$
$$a_2 m \equiv 1 \pmod{t - Km},$$
and
$$a_3\{(K+1)m - t\} \equiv 1 \pmod{m}.$$

In the notation of (2.1),

$$a_1 m = 1 + b_1 t, \quad a_1 < t, \quad b_1 < m ;$$
$$a_2 m = 1 + b_2(t - Km), \quad a_2 < t - Km, \quad b_2 < m ;$$
$$a_3(Km + m - t) = 1 + b_3 m, \quad a_3 < m, \quad b_3 < Km + m - t.$$

We have $\pmod{m}$

$$b_1 t \equiv -1, \quad b_2 t \equiv -1 \quad \text{and} \quad a_3 t \equiv -1 ;$$

and since all are least positive residues,

$$b_1 = b_2 = a_3.$$

From the first two equations, by subtraction,

$$(a_1 - a_2)\,m = b_2 Km$$
$$= a_3 Km.$$

Hence $a_1 = a_2 + Ka_3$, that is,

$$\frac{1}{m} \pmod t = \frac{1}{m} \pmod{\overline{t - Km}} + K \left\{ \frac{1}{Km + m - t} \pmod m \right\}. \tag{3.3}$$

Equations (3.1), (3.2), (3.3) enable us easily to construct the following table of values of a:

$t=$	1	2	3	4	5	6	7	8	9	10	11	12
$m=$ 1	1	1	1	1	1	1	1	1	1	1	1	1
2	1		2		3		4		5		6	
3	1	1		3	2		5	3		7	4	
4	1		1		4		2		7		3	
5	1	1	2	1		5	3	5	2		9	5
6	1				1		6				2	
7	1	1	1	3	3	1		7	4	3	8	7
8	1		2		2		1		8		7	
9	1	1		1	4		4	1		9	5	
10	1		1				5		1		10	
11	1	1	2	3	1	5	2	3	5	1		11
12	1				3		3				1	

(3.1) is reflected in the repetition of a cycle in each vertical column. (3.2) is the relation between the two halves of each cycle. By (3.3), in any horizontal row each group is formed from the preceding one by the addition of constant differences which are the elements, reading upwards, of the column $t=m$. Thus when $m=5$, the groups 5, 3, 5, 2; 9, 5, 8, 3, etc., are formed from the first group 1, 1, 2, 1 by addition of constant differences 4, 2, 3, 1 from the column $t=5$. It will be noted that by virtue of (3.1) the upward diagonal and horizontal row (first group) which start from any value of m are identical; the table may therefore be stopped at any diagonal.

If equations (3.1), (3.3) are used to evaluate Σa for a particular m, the calculation of the K's is practically equivalent to the H.C.F. process required for (2.1), but at least we avoid the construction of a/b as the penultimate convergent of the continued fraction for t/m.

4. G. B. Mathews gives (*Theory of Numbers*, 318) the residues of $(p-1)!+1 \pmod{p^2}$ for primes up to 41. We tabulate below the residues of $\Sigma a \pmod m$ for values of m from 3 to 80:

		2	20	7	32	19	55	62
		8	6	14	16	16	56	40
$(m=3)$	2	0	16	5	26	19	41	22
	2	0	8	18	28	40	14	8
	0	9	10	30	33	11	3	11
	2	14	16	4	2	28	48	24
	3	10	21	18	12	21	24	44
	6	4	0	24	32	48	52	74
	3	4	7	1	38	54	49	53
$(m=10)$	8	8	14	28	44	44	4	68

C. G. PARADINE.

1541. *The sums of powers of the natural numbers.*

The usual elementary method of obtaining the formula for $S_p(n)$, the sum of the pth powers of the first n natural numbers, is by "differencing" some polynomial function of n, generally n^{p+1} or perhaps $n(n+1)\dots(n+p)$. We can get the formulae more rapidly if we difference $\{n(n+1)\}^r$ for the sums of odd powers S_{2r-1} and $(2n+1)\{n(n+1)\}^r$ for the sums of even powers S_{2r}. Thus

$$\{n(n+1)\}^r - \{n(n-1)\}^r = 2n^r\{{}^rC_1\, n^{r-1} + {}^rC_3\, n^{r-3} + \dots\}.$$

Summing from 1 to n we get

$$\{n(n+1)\}^r = 2\{{}^rC_1\, S_{2r-1} + {}^rC_3\, S_{2r-3} + \dots\}, \quad \dots\dots\dots\dots\dots\dots(1)$$

the sum on the right ending with $2S_r$ or $2rS_{r+1}$ according as r is odd or even. Evidently $r=1, 2$ give the elementary formulae

$$S_1 = \tfrac{1}{2}n(n+1), \qquad S_3 = \tfrac{1}{4}n^2(n+1)^2.$$

Putting $r=3, 4, 5$, say, and writing $N \equiv n(n+1)$ for brevity, we have

$$N^3 = 6S_5 + 2S_3, \qquad N^4 = 8S_7 + 8S_5,$$
$$N^5 = 10S_9 + 20S_7 + 2S_5,$$

which give without difficulty

$$S_5 = \tfrac{1}{6}N^3 - \tfrac{1}{12}N^2, \qquad S_7 = \tfrac{1}{8}N^4 - \tfrac{1}{8}N^3 + \tfrac{1}{12}N^2,$$
$$S_9 = \tfrac{1}{10}N^5 - \tfrac{1}{4}N^4 + \tfrac{3}{10}N^3 - \tfrac{3}{20}N^2.$$

The general formula begins

$$S_{2r+1} = \frac{N^{r+1}}{2(r+1)} - \frac{r-1}{12}N^r + \frac{r(r-2)(7r-1)}{720}N^{r-1} - \dots,$$

but the form of the coefficients rapidly becomes obscure. We should note, however, that every S_{2r-1}, except of course S_1, has the factor N^2.

For the even powers we have

$$(2n+1)\{n(n+1)\}^r - (2n-1)\{n(n-1)\}^r$$
$$= 2n^r\{(2r+1)n^{2r} + \frac{2r-1}{3}\,{}^rC_2\, n^{2r-2} + \dots\},$$

and, on summation,

$$(2n+1)\{n(n+1)\}^r = 2\{(2r+1)S_{2r} + \frac{2r-1}{3}\,{}^rC_2\, S_{2r-2} + \dots\}, \quad \dots(2)$$

the last term being $2(r+2)S_{r+1}$ or $2S_r$ according as r is odd or even.

Evidently $r=1$ gives the elementary result

$$S_2 = \tfrac{1}{6}n(n+1)(2n+1).$$

More generally, taking $r=2, 3, 4$, say, we get

$$(2n+1)N^2 = 10S_4 + 2S_2, \qquad (2n+1)N^3 = 14S_6 + 10S_4,$$
$$(2n+1)N^4 = 18S_8 + 28S_6 + 2S_4,$$

and so

$$S_4=(2n+1)(\tfrac{1}{10}N^2-\tfrac{1}{30}N),\quad S_6=(2n+1)(\tfrac{1}{14}N^3-\tfrac{1}{14}N^2+\tfrac{1}{42}N),$$
$$S_8=(2n+1)(\tfrac{1}{18}N^4-\tfrac{1}{9}N^3+\tfrac{1}{10}N^2-\tfrac{1}{30}N).$$

The general formula begins

$$S_{2r}=\frac{2n+1}{2r+1}\left\{\frac{N^r}{2}-\frac{r(r-1)N^{r-1}}{12}+\frac{r(r-1)(r-2)(7r-1)}{720}N^{r-2}-\ldots\right\}.$$

Here S_{2r} always contains the factor $(2n+1)N$.

Comparison of the formulae for S_{2r}, S_{2r+1} suggests that, if in these formulae we replace the integer n by a variable x, we shall have generally

$$(2r+1)S_{2r}(x)=\frac{d}{dx}S_{2r+1}(x).$$

This can be established inductively by comparing the fundamental formulae (1), (2). It is also an immediate consequence of the expression of $S_p(x)$ in terms of Bernoulli's numbers *

$$(r+1)S_r(n-1)=\frac{n^{r+1}}{r+1}-\tfrac{1}{2}(r+1)n^r+{}^nC_2B_1n^{r-1}-{}^nC_4B_2n^{r-3}+\ldots,$$

in which the terms n^{r-2}, n^{r-4}, ... are absent.

With this evidence before us we can now construct a method for establishing the formulae that, stated without preface, might appear artificial. Beginning with the odd powers we write

$$f_r(x)=\frac{\{x(x+1)\}^{r+1}}{2(r+1)}+C_1\{x(x+1)\}^r+\ldots+C_{r-1}\{x(x+1)\}^2,\quad\ldots(3)$$

and choose the $r-1$ " undetermined coefficients " $C_1, \ldots, C_{r-1}$ so that the $r-1$ odd powers $x^{2r-1}, \ldots, x^3$ disappear. This leaves $f_r(x)$ in the form

$$f_r(x)=\tfrac{1}{2}x^{2r+1}+\text{a sum of even powers,}\quad\ldots\ldots\ldots\ldots\ldots(4)$$

and so $$f_r(x)-f_r(-x)=x^{2r+1}.$$

But from its form (3),

$$f_r(-x)=f_r(x-1).\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Thus we have the difference formula

$$f_r(x)-f_r(x-1)=x^{2r+1},\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

which with $f_r(0)=0$ at once gives

$$f_r(n)=S_{2r+1}(n).$$

Now by differentiating (4) we have

$$\frac{1}{2r+1}f_r'(x)=\tfrac{1}{2}x^{2r}+\text{a sum of odd powers,}$$

and so $$\frac{1}{2r+1}\{f_r'(x)+f_r'(-x)\}=x^{2r}.$$

* See, for instance, E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge, 1920, chap. 7, § 7·2, 127.

But differentiation of (5) gives

$$f_r'(-x) = -f_r(x-1),$$

and so we have the difference formula

$$\frac{1}{2r+1}\{f_r'(x) - f_r'(x-1)\} = x^{2r},$$

which we could have got at once by differentiating (6). This with $f'(0) = 0$ at once gives

$$\frac{f_r'(n)}{2r+1} = S_{2r+1}(n).$$

As a simple illustration of the method let us recalculate S_9 directly. We write

$$S_9 = \tfrac{1}{10}n^5(n+1)^5 - C_1 n^4(n+1)^4 + C_2 n^3(n+1)^3 - C_3 n^2(n+1)^2.$$

To make the terms n^7, n^5, n^3 disappear we have

$$1 - 4C_1 = 0, \quad \tfrac{1}{10} - 4C_1 + 3C_2 = 0, \quad C_2 - 2C_3 = 0,$$

which give $C_1 = \frac{1}{4}, \quad C_2 = \frac{3}{10}, \quad C_3 = \frac{3}{20},$

as before. T. W. C.

1542. *If* $y^3 + 3xy + 2x^3 = 0$, *then*

$$x^2(1+x^3)\frac{d^2y}{dx^2} - \tfrac{3}{2}x\frac{dy}{dx} + y = 0.$$

Readers who have written pages of algebra—and which of us has not?—round this innocent-looking Tripos question, which is in *Pure Mathematics* as an exercise, may be amused by the following solution.

Substitute $x = -u/v$, $y = -1/v$, and use u as independent variable. Then

$$3v = 2u^2 + \frac{1}{u}, \quad 3v' = 4u - \frac{1}{u^2}, \quad 3v'' = 4 + \frac{2}{u^3} = \frac{6v}{u^2};$$

$$x' = \frac{1}{v^2}(uv' - v) = \frac{2}{v^2}\left(u^2 - \frac{1}{u}\right), \quad y' = \frac{1}{v^2}\cdot v';$$

$$\frac{dy}{dx} = \frac{v'}{v^2x'}, \quad \frac{d^2y}{dx^2} = \frac{1}{v^4x'^3}\begin{vmatrix} uv'-v, & uv'' \\ v', & v'' \end{vmatrix} = -\frac{v''}{v^3x'^3} = -\frac{2}{u^2v^2x'^3};$$

also $$27(u^3v^3 - u^6) = (2u^3+1)^3 - 27u^6 = (u^3-1)^2(8u^3+1).$$

Hence

$$x^2(1+x^3)\frac{d^2y}{dx^2} - \frac{3}{2}x\frac{dy}{dx} + y = y\left\{\frac{2(1+x^3)}{v^3x'^3} - \frac{3uv'}{2v^2x'} + 1\right\}$$

$$= y\left\{\frac{8u^3+1}{4(u^3-1)} - \frac{3(4u^3-1)}{4(u^3-1)} + 1\right\}$$

$$= 0.$$

Needless to say, this verification bears no relation to the principles on which the result depends; it is offered as entertainment, not as instruction. E. H. N.

1543. *The expression of an arbitrary elliptic function in terms of the Weierstrassian function.*

In accordance with a general theorem due to Liouville, any elliptic function $f(z)$ can be expressed in the form $F(\wp z)+\wp' z\,G(\wp z)$, where $F(u)$, $G(u)$ are rational functions of their argument. To reach this result, $f(z)$ is written as the sum $\phi(z)+\theta(z)$, where $\phi(z)$ is the even function $\frac{1}{2}\{f(z)+f(-z)\}$ and $\theta(z)$ is the odd function $\frac{1}{2}\{f(z)-f(-z)\}$; since $\theta(z)/\wp' z$ is even, the fundamental theorem is that *An even elliptic function $\phi(z)$ is a rational function $\Phi(\wp z)$ of $\wp z$,* identity of periods being of course presumed.

The most illuminating proof of this theorem constructs the function $\Phi(\wp z)$ from the zeros and poles of $\phi(z)$. However the proof is arranged, it is necessary to discriminate against the lattice points if they are zeros or poles. The argument in Copson's *Functions of a Complex Variable* runs briefly as follows. Suppose first that $\phi(z)$ is neither zero nor infinite at the origin. If a is a zero of order m, so also is $-a$; if b is a pole of order n, so also is $-b$; also $\Sigma m=\Sigma n$. Hence $\Pi(\wp z-\wp a)^m/\Pi(\wp z-\wp b)^n$, with the products embracing only one of the two points $\pm a$ and only one of the two points $\pm b$ in each typical pair, is a function with the periods and the structure of $\phi(z)$, and can be identified with $\phi(z)/\phi(0)$. That is to say,

$$\Phi(\wp z)=\phi(0)\,\Pi(\wp z-\wp a)^m/\Pi(\wp z-\wp b)^n.$$

Now to remove the restriction regarding the origin, note that since the zeros and poles away from the lattice points can be paired, the order of the origin as zero or pole is an even number $2q$; then $\wp^q z\phi(z)$ in the one case and $\wp^{-q}z\phi(z)$ in the other case is an even function $\psi(z)$ neither zero nor infinite at the origin, and expressible therefore as a rational function $\Psi(\wp z)$ of $\wp z$, whence the original function $\phi(z)$ is $\wp^{-q}z\,\Psi(\wp z)$ or $\wp^q z\,\Psi(\wp z)$.

Logically this method of dealing with a zero or pole at the origin is irreproachable; nevertheless it is, in a very strict sense, absurd. For the function $\wp z$ has two simple zeros $\pm c$. What has become of them? In the analysis of the product $\wp^{\pm q}z\phi(z)$, they imply that the function $\Psi(\wp z)$ has, in addition to the factors occasioned by the zeros and poles of $\phi(z)$, the factor $(\wp z-\wp c)^{\pm q}$, that is, $\wp^{\pm q}z$, and when we divide or multiply by $\wp^q z$ to complete the determination of $\Phi(\wp z)$ from $\Psi(\wp z)$, the factor introduced gratuitously disappears automatically.

In other words, if we attend to the zeros and poles which are not at the origin, the origin, if not an ordinary point, looks after itself, whether it is a zero or a pole. For example, a function $\phi(x)$ whose only zeros and poles away from the origin are double zeros at $\pm a$ and simple poles at $\pm b$ is a constant multiple of $(\wp z-\wp a)^2/(\wp z-\wp b)$.

Since away from the origin the order computed in terms of zeros exceeds by two the order computed in terms of poles, the origin is a double pole, but this double pole is supplied *automatically* by the quotient $(\wp z - \wp a)^2/(\wp z - \wp b)$. If we argue in terms of the function $\phi(z)/\wp z$ which is neither zero nor infinite at the origin, we have to recognise that this quotient has the double zeros of $\phi(z)$ at $\pm a$, the simple poles of $\phi(z)$ at $\pm b$, *and the simple poles of* $1/\wp z$ *at* $\pm c$, and appears therefore as a multiple of $(\wp z - \wp a)^2/(\wp z - pb)(\wp z - \wp c)$, that is, of $(\wp z - \wp a)^2/(\wp z - \wp b)\wp z$; to express $\phi(z)$ we have now to drop the factor $\wp z$ from the denominator, and it is clear that we have gained nothing whatever by introducing it.

In *Modern Analysis* the procedure is described correctly : " If any one of the points a or b is congruent to the origin, we omit the corresponding factor $\wp z - \wp a$ or $\wp z - \wp b$. The zero (or pole) of the product and the zero (or pole) of $\phi(z)$ at the origin are then of the same order of multiplicity." It may fairly be objected that the ellipsis is excessive, since the a's and b's are paired and the origin must stand alone, and that the conclusion needs a word of justification. The explanation is given most simply by Jordan in his *Cours d'Analyse*. If the functions $\phi(z)$ and $\Phi(\wp z)$ have the same structure away from the origin, the origin is the only point in the cell which can be *either* a zero *or* a pole of the quotient $\phi(z)/\Phi(\wp z)$, and since an elliptic function which is not a constant has *both* zeros *and* poles, the quotient is constant. The connoisseur of obscurities should read the version of this argument in Henry's *Fonctions Elliptiques*.

A more serious criticism may be brought against the proof that has been outlined. The half-period values $\omega_1, \omega_2, \omega_3$ also need special consideration, and they cannot be said to have received it. The point ω_r is a double zero of $\wp z - \wp\omega_r$, and if in accordance with instructions we include a factor $(\wp z - \wp\omega_r)^m$ in the numerator or a factor $(\wp z - \wp\omega_r)^n$ in the denominator, we provide the function with a zero of order $2m$ or a pole of order $2n$. To provide the function with a zero or a pole of the prescribed order m or n, we need only the factor $(\wp z - \wp\omega_r)^{m/2}$ or $(\wp z - \wp\omega_r)^{n/2}$, and although this factor is an elliptic function even if m or n is odd, its periods then are not those of $\wp z$ and it is not a polynomial in $\wp z$. To rehabilitate the proof we must have the lemma, which Jordan does not omit : *If* ω_r *is a zero or a pole of an even elliptic function, it is a zero or pole of even order.* This lemma is a corollary of a much more interesting theorem. Let 2ω be a period of an ODD elliptic function $\theta(z)$. Then unless ω is a pole of the function, $\theta(-\omega) = \theta(\omega)$ because 2ω is a period and $\theta(-\omega) = -\theta(\omega)$ because $\theta(z)$ is odd. Hence

Every half-period of an odd elliptic function is either a pole or a zero.

COROLLARIES :

(1) The smallest value of s for which the sth derivative of an odd function $\theta(z)$ or of an odd function $1/\theta(z)$ is not zero when $z = \omega$ is necessarily odd, and therefore the zero or pole of $\theta(z)$ is of odd order.

(2) The smallest value of s for which the sth derivative of an even

function $\phi(z)$ or of an even function $1/\phi(z)$ is not zero when $z=\omega$ is necessarily even ; hence the lemma used above.

(3) If a function of the third order has a triple pole at the origin, the three points ω_1, ω_2, ω_3 can function as required only if each of them is a simple zero ; hence the structure of $\wp' z$.

(4) An odd function of the second order has two of the four points 0, ω_1, ω_2, ω_3 for simple poles and the other two for simple zeros. As is to be expected, there are six functions answering to this description ; they are essentially Jacobian functions, or logarithmic derivatives of Jacobian functions, according to the lattices with which they are associated.

E. H. N.

1544. *The Child's Arithmetic. Mdcccxxxvii.*

The Child's Arithmetic, **A Manual of Instruction** for the NURSERY and INFANT SCHOOLS, is of some interest at the present time when Projects are becoming a feature of Arithmetic textbooks in this country. The book consists mainly of thirty-two LESSONS, each of which, though rarely a Project, yet resembles one, in that a picture with the descriptive matter beneath provides an atmosphere for the sums. The LESSONS are all, according to the headings, ABOUT something, DOGS, LIONS, A CAT AND HER KITTENS, SCALES AND WEIGHING, RATS EATING NUTS, MONEY, A POOR WOMAN AND HER HONEY, TRAVELLING, COTTON, HAY AND CORN, CLOCKS AND WATCHES. They cover the first four rules and the measures quite simply, though occasionally there is a difficult sum, while the wording of a sum, except in the first third of the book, may lack conciseness and may be partly irrelevant.

> " It is now spring, Miss Susan, and all the birds are singing, and the flowers budding in the warm sunshine. You wish to play in the garden, and your pelisse and stuff gown are too hot. Your mother therefore gets 4 yards of muslin at a shilling a yard . . . ."

This is one of the extreme examples ; generally the irrelevancies are adjectives or short phrases—" poor old woman ", " a rich neighbour ", " whose name was Johnny Black ", " who was inclined to be a dandy ". Apart from these features there is little to date the sums except occasionally a price or a material. Many are quite up to date :

> " If a boy has 45 marbles, and gives away 16 of them ; how many has he left?
>
> A woman used 6 lbs. of tea in a year ; how many ounces did she use in a month?
>
> A farmer had 5 cows, which ate 27 bushels of potatoes ; how many did each cow eat? "

It is worth noting that the fractions, half, quarter, third, are used, very simply, and not in connection with the four rules except in one case, $1\frac{3}{4}$ bushels $\times 40$.

There are also four shorter LESSONS at the end of the book in which ADDITION, SUBTRACTION, MULTIPLICATION, and DIVISION are illustrated, pictorially and verbally, and explained. The last page is partly missing, but it appears that "he" was fast asleep when something happened. He knew that the offender was one of 8 boys, but, not discovering the culprit, "he divided the 40 lashes among them." "How many did each boy receive?" This problem helps to show us, what the irrelevancies mentioned above suggest, that the purpose of a LESSON is not merely arithmetical. Occasionally the moral teaching is direct, and apart from the sums. "Children, whose parents are poor, should not be too anxious for new clothes, . . . and rich people should not dress their children in an extravagant manner, for, in doing so, they train them up to vanity." "Some squirrels are killed by mischievous boys." One picture, a boy, amidst barrels, drawing beer, reminds me of a problem I once set in a draft examination paper. It divided the Examination Board into two parties, teetotal and otherwise, and argument became hot and furious. It looked as though a cherished problem would have to be deleted, until a timely suggestion saved the situation. The word "ginger" was inserted before "beer", and temptation thus removed from innocent scholarship candidates. The method was different a hundred years ago.

> "Milk is wholesome and pleasant to drink, and so is tea and coffee; but gin, whisky, and other kinds of spirits are not wholesome, and many persons injure themselves and their families by drinking them. There was once a working man who received 20 shillings of wages every week, and out of that 20 shillings he regularly bought every week 3 shillings worth of whisky, which he drank.... Now, tell me, how many pounds did this foolish man spend during a whole year on this very bad indulgence in liquor."

The Project, as a form of educational problem, originated in the United States of America, and it is interesting to note that *The Child's Arithmetic* originally appeared there, where it was "eminently successful". It was therefore altered considerably "to adapt it to the habits and feelings of a British community", and was published in May, 1837, by William S. Orr & Co., Paternoster Row, London. It contains 64 pages, 6½ in. × 4 in., and has frontispiece on both title-page and paper back. R. S. WILLIAMSON.

1545. *The use of the word "scalar".*

At one time a "scalar" used to be explained in textbooks as a signless number, but that is not the present-day usage. The scalar product of two vectors may be positive or negative, and so may the scalar part of a quaternion.

So scalars appear to be the usual positive or negative (real) numbers of our ordinary elementary algebra. In fact we introduce scalars to the school-boy under the name of "*directed numbers*".

Later on, of course, to distinguish them from vectors, we say that *scalars are not associated with a direction.*

As Walt Whitman remarked: "Do I contradict myself? Very well then, I contradict myself." C. O. TUCKEY.

1546. *A proof of Murphy's formula.*

Murphy's formula is

$$\frac{1}{n!}\frac{d^n}{dx^n}(x\log x)^n = 1 + s_1\log x + \frac{s_2}{2!}(\log x)^2 + \ldots + \frac{s_n}{n!}(\log x)^n,$$

where s_r is the sum of the products, r at a time, of the numbers $1, 2, \ldots, n$.

To prove this, let us set

$$x = e^t, \quad t = \log x,$$

and let D stand for d/dt.

Then

$$\begin{aligned}\frac{1}{n!}\frac{d^n}{dx^n}(x\log x)^n &= \frac{1}{n!}\frac{x^n}{x^n}\frac{d^n}{dx^n}(x\log x)^n \\ &= \frac{1}{n!}\cdot\frac{1}{e^{nt}}\cdot D(D-1)(D-2)\ldots(D-n+1)\,.\,e^{nt}t^n \\ &= \frac{1}{n!}\frac{e^{nt}}{e^{nt}}\cdot(D+n)(D+n-1)\ldots(D+1)t^n \\ &= \frac{1}{n!}(D^n + s_1D^{n-1} + s_2D^{n-2} + \ldots + s_n)t^n \\ &= 1 + s_1t + \frac{s_2}{2!}t^2 + \ldots + \frac{s_n}{n!}t^n \\ &= 1 + s_1\log x + \frac{s_2}{2!}(\log x)^2 + \ldots + \frac{s_n}{n!}(\log x)^n.\end{aligned}$$

KUO HUAN-TING.

1547. *Pascal's theorem.*

With reference to Note 1511, the following elegant analytical proof of Pascal's theorem is not perhaps so well known as it deserves to be.

$S=0$ is a conic; $\alpha=0$, $\beta'=0$, $\gamma=0$, $\alpha'=0$, $\beta=0$, $\gamma'=0$ are the sides of an inscribed hexagon taken in order. Then, if L be linear, $LS=\alpha\beta\gamma$ is a cubic through the vertices of the hexagon. If the three constants of L be properly chosen, it may be *any* cubic through these six points: in particular it may be equivalent to $\alpha'\beta'\gamma'=0$. Then $\alpha\beta\gamma=0$, $\alpha'\beta'\gamma'=0$ must intersect either on S or on L. The vertices of the hexagon account for the six intersections with S; the three others must lie on L. That is, $\alpha, \alpha' : \beta, \beta' : \gamma, \gamma'$ meet in points which lie on the straight line $L=0$. PERCY J. HEAWOOD.

1548. *Analytical proof of Pascal's theorem.* (*See* Note 1511, xxv, p. 109.)

If ABC is the triangle of reference, and P, Q, R are any three points in the plane, the lines CQ, BR are $x/x_Q = y/y_Q$, $x/x_R = z/z_R$, and they meet in $(x_Q x_R,\ y_Q x_R,\ x_Q z_R)$. Pascal's theorem is the identity

$$\begin{vmatrix} x_Q x_R, & y_Q x_R, & x_Q z_R \\ y_R x_P, & y_R y_P, & z_R y_P \\ x_P z_Q, & z_P y_Q, & z_P z_Q \end{vmatrix} = \begin{vmatrix} y_P z_P, & z_P x_P, & x_P y_P \\ y_Q z_Q, & z_Q x_Q, & x_Q y_Q \\ y_R z_R, & z_R x_R, & x_R y_R \end{vmatrix}.$$

(Compare C. Smith, *Conic Sections*, p. 299; A. C. Jones, *Algebraical Geometry*, p. 490. A clumsy version of the same proof is given by N. M. Ferrers and C. J. C. Price in their textbooks on trilinear coordinates.) E. H. N.

1549. *Note on some definitions in navigation.*

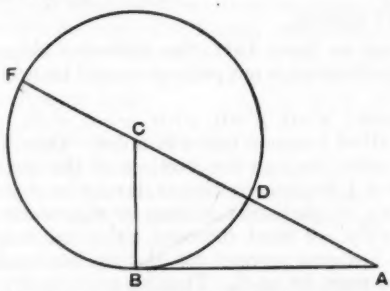
It would be an improvement if the Air Council could see their way to alter some of the definitions in A.P. 1234, quoted on page 69 of the *Gazette*.

The (true) *course* of an aircraft is the angle from the (true) north to the direction in which the axis of the aircraft is pointing.

By making the change from " angle between " to " angle from . . . to ", they will not only help their embryo navigators, but at the same time will set a good example to professional teachers of mathematics. The same change is applicable to magnetic and compass courses and to tracks and bearings. A. R.

1550. *A simpler approach to Pythagoras' theorem.*

Euclid's proof of Pythagoras' theorem is usually found somewhat complicated by the beginner. There are many similar proofs, for instance, that due to Bhaskara.* In addition to these, the following proof, based upon circle work, has been found quite popular among matriculation forms. This proof would necessitate taking the circle before Pythagoras—the reverse of the usual procedure. The ease with which this proof is assimilated, however, seems to justify the small alteration beyond all doubt.



* Sir T. L. Heath, *The Thirteen Books of Euclid's Elements*, I, p. 355.

The triangle ABC is right-angled at B. Draw the circle centre C and radius BC. Produce AC to cut the circle again at F.

By the proportionality of the sides of the similar triangles ABD and ABF, we have

$$\begin{aligned} AB^2 &= AD \,.\, AF \\ &= (AC - CD)(AC + CF) \\ &= (AC - BC)(AC + BC) \\ &= AC^2 - BC^2. \end{aligned}$$

Thus $$AC^2 = AB^2 + BC^2.$$

D. J. PRICE (per Miss E. A. G. KNOWLES).

In teaching elementary geometry, I place work on similar triangles immediately after the "equal intercept" theorem and prove the intersecting chords theorem, the secant theorem, and the tangent and secant theorem by the methods of similar triangles. My pupil, J. S. Hunt, has suggested to me that Pythagoras' theorem can then be proved rigorously by the method given above by Miss Knowles. This provides additional evidence for her remark that pupils appreciate this order of ideas. A. G. CARPENTER.

1551. *Sign of the perpendicular from a point to a line.*

The accepted rule for the sign of the perpendicular from a point (x', y') to a straight line $ax + by + c = 0$ depends on comparing it with the perpendicular from the origin. Probably the best form of the rule for working purposes is as follows:

Make c positive and use the formula,

$$\text{perpendicular} = \frac{ax' + by' + c}{\sqrt{(a^2 + b^2)}};$$

if the result comes out positive, the point is on the same side of the line as the origin; if it comes out negative, the point is on the opposite side.

This rule has the disadvantage that it does not apply to the case when the straight line passes through the origin; in such a case there is not even a convention as to the sign of the perpendicular.

The alternative rule worked out below is free from these objections.

Preliminary considerations.

1. By the perpendicular from a point to a line, we mean the distance from the point to the foot of the perpendicular; the distance from the foot to the point would be minus the former distance.

2. In fact, just as we have a convention of signs for distances along OX or OY, so we must have a convention of signs for any straight line in the plane.

3. Just as it is found convenient to give all parallels to Ox and Oy the same positive sense (from left to right and upwards respectively), so it will be convenient to give all parallel lines (in any

direction) the same positive sense. It will therefore be sufficient to assign a positive sense to every straight line through the origin.

4. (*a*) If the line has a positive gradient, its positive sense must be taken to point into the first quadrant, otherwise when the line takes up the position Ox or Oy, the positive senses of these would come opposite to those usually taken (see Fig. 1).

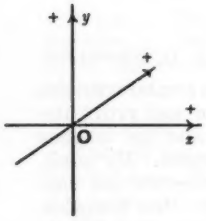


FIG. 1.

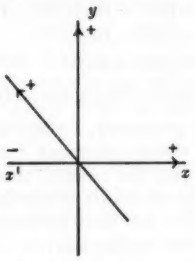


FIG. 2.

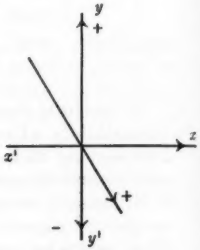


FIG. 3.

(*b*) If the line has a negative gradient, its positive sense may be taken, indifferently, to point either into the 2nd or the 4th quadrant (see Figs. 2 and 3). In the first case the extreme position Ox' must be excluded, being already included in Ox; in the second case the extreme position Oy' must be excluded, being already included in Oy.

5. If we adopt the convention of polar coordinates, and take the positive sense of a line inclined at an angle θ to Ox along the other arm of the angle (Ox being one arm), we may say either that all directions in the plane are included in the range $0°$ to $180°$ (excluding the last) or that all directions are included in the range $-90°$ to $+90°$ (excluding the first).

Working out of the rule.

Coming now to the problem of the sign of the perpendicular from a point to a line, consider first the case where the line has a negative gradient—and the perpendicular therefore a positive gradient (Fig. 4). (The line may or may not pass through the origin.) The perpendicular from the point to the line is, according to our convention, positive if the point is below the line, and negative if above.

Next consider the case where the line has a positive gradient—and the perpendicular therefore a negative gradient. If we regard the inclination of the perpendicular as lying between $90°$ and $180°$ (*i.e.* the positive sense pointing into the 2nd quadrant), the perpendicular from the point to the line will be positive if the point is below the line and negative if above (see Fig. 5). This agrees with the convention arrived at for lines with a negative gradient. But if we regard the inclination of the perpendicular as lying between $-90°$ and $0°$ (*i.e.* the positive sense pointing into the 4th quadrant) the perpendicular will be positive if the point is above

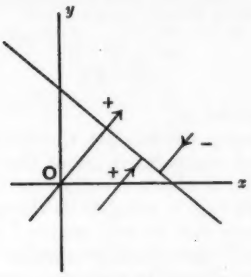


FIG. 4.

the line, and negative if below (Fig. 6); which is the opposite convention to that arrived at for a line with a negative gradient.

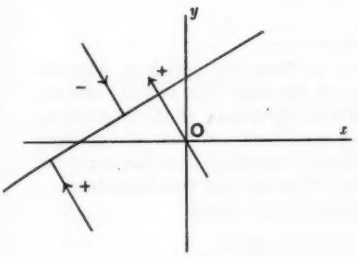


FIG. 5.

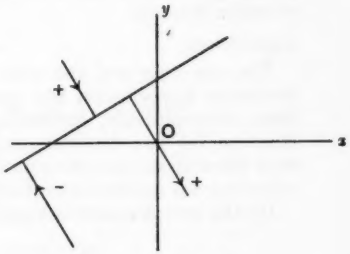


FIG. 6.

Plainly it will be simplest to have the same convention for all straight lines. Hence we adopt the convention that all directions are included in the range 0° to 180° (excluding the latter), and that a perpendicular from a point to a line is positive if the point is below the line and negative if above.

Now suppose the line be $y - mx - c = 0$, the numerical value of the perpendicular may be shown to be

$$\frac{y' - mx' - c}{\pm\sqrt{(1+m^2)}}.$$

To get the algebraic value, that sign (+ or −) must be taken, which will give a sign to the perpendicular in agreement with the convention we have arrived at; *i.e.* the result must come out positive if (x', y') is below the line, and negative if above.

Now $y' - mx' - c$ is positive if (x', y') is above the line, and negative if below. Therefore the negative sign must be taken in the denominator. That is, the algebraic value of the perpendicular is

$$\frac{-y' + mx' + c}{\sqrt{(1+m^2)}},$$

or more generally, (for a line $ax+by+c=0$),

$$\frac{ax'+by'+c}{\sqrt{(a^2+b^2)}},$$

where b is made negative (α).

Conversely, if on using the rule (α), the result comes out positive, we may conclude that the point is below the line; and if it comes out negative, the point is above the line.

The case where the line is parallel to the y-axis requires special consideration, for there can be no "above" or "below" the line. The usual convention of signs for lines parallel to the x-axis demands that the perpendicular should be reckoned positive if the point is to the left of the line and negative if it is to the right. Hence if the equation be $x-a=0$, the perpendicular from (x', y') must be written $-x'+a$; *i.e.* the equation must be written $-x+a=0$, or more generally $-ax+b=0$, with the x-term negative, before the substitution is made.

Application.

The new rule and the convention of signs on which it depends obviously apply when the line passes through the origin. Hence there is no difficulty in dealing with the problem of distinguishing between the two bisectors of the angles between two given lines, even when these pass through the origin; in which case the accepted rule does not enable us to distinguish between the two bisectors.

By the new rule and convention, the bisectors are

$$\frac{ax+by+c}{\sqrt{(a^2+b^2)}}=\pm\frac{a'x+b'y+c'}{\sqrt{(a'^2+b'^2)}},$$

where b, b' are made negative; the positive sign giving the bisector of the angle in which points are above (or below) both lines, and the negative sign the bisector of the angle in which points are above one and below the other.

C. BLACK.

1552. *Formulae for curvature of plane curves.*

The following arrangements of proofs of the elementary formulae for the curvature of plane curves may be well known, but I do not recollect having seen them in any of the standard textbooks. The sections 2·1, 2·2 were suggested to me by my pupils, Constructor Lieutenants R. J. Daniel and C. P. Oldridge respectively.

1. The unit tangent and normal vectors are denoted by $\mathbf{T}$, $\mathbf{N}$; p is the perpendicular OY from the origin to the tangent at P, q is the intercept YP. We assume the fundamental formulae for plane curves:

$$d\mathbf{r}/ds=\mathbf{T},\quad d\mathbf{T}/d\psi=\mathbf{N},\quad d\mathbf{N}/d\psi=-\mathbf{T}.$$

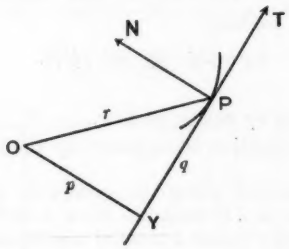
2·1. If $\mathbf{k}$ is a unit vector perpendicular to the plane,

$$p\mathbf{k}=\mathbf{r}\wedge\mathbf{T},$$

and differentiating with respect to ψ we have

$$(dp/d\psi)\mathbf{k} = \rho(\mathbf{T} \wedge \mathbf{T}) + \mathbf{r} \wedge \mathbf{N}$$
$$= \mathbf{r} \wedge \mathbf{N}, \quad \text{.....(i)}$$
$$(d^2p/d\psi^2)\mathbf{k} = \rho(\mathbf{T} \wedge \mathbf{N}) - \mathbf{r} \wedge \mathbf{T}$$
$$= (\rho - p)\mathbf{k}.$$

Hence $$\rho = p + d^2p/d\psi^2.$$



Incidentally from (i)

$$(dp/d\psi)\mathbf{k} = \mathbf{r} \wedge \mathbf{N} = q\mathbf{k},$$

and so $$q = dp/d\psi.$$

2·2. $$\dot{r} = -p\mathbf{N} + q\mathbf{T}.$$

Thus $$d\mathbf{r}/d\psi = p\mathbf{T} - (dp/d\psi)\mathbf{N} + q\mathbf{N} + (dq/d\psi)\mathbf{T}.$$

But $$d\mathbf{r}/d\psi = \rho\mathbf{T}.$$

Hence $$0 = \left(p + \frac{dq}{d\psi} - \rho\right)\mathbf{T} + \left(q - \frac{dp}{d\psi}\right)\mathbf{N},$$

whence $$q = dp/d\psi, \quad \text{.....(ii)}$$

and $$\rho = p + dq/d\psi \quad \text{.....(iii)}$$
$$= p + d^2p/d\psi^2, \quad \text{by (ii).}$$

3. Since $$r^2 = \mathbf{r}^2,$$
$$r\,dr/d\psi = \mathbf{r}\,d\mathbf{r}/d\psi$$
$$= (\mathbf{r}\mathbf{T})\rho$$
$$= q\rho$$
$$= \rho\,dp/d\psi \quad \text{by (ii).}$$

Hence $$\rho = r\,dr/dp.$$

This also follows by differentiating $r^2 = p^2 + q^2$ with respect to ψ and using (ii) and (iii).

4. Since $$d\mathbf{r}/ds = \mathbf{T},$$

we have $$d^2\mathbf{r}/ds^2 = d\mathbf{T}/ds = \mathbf{N}/\rho;$$

thus
$$\frac{d\mathbf{r}}{ds}\wedge\frac{d^2\mathbf{r}}{ds^2}=(\mathbf{T}\wedge\mathbf{N})/\rho$$
$$=\mathbf{k}/\rho,$$
and so
$$\frac{1}{\rho}=\frac{dx}{ds}\cdot\frac{d^2y}{ds^2}-\frac{dy}{ds}\cdot\frac{d^2x}{ds^2}.$$

If x, y are given as functions of a parameter t, then
$$dx/ds=\dot{x}/\dot{s},\quad d^2x/ds^2=(\dot{s}\ddot{x}-\ddot{s}\dot{x})/\dot{s}^3,$$
and on substitution we have
$$1/\rho=(\dot{x}\ddot{y}-\ddot{x}\dot{y})/(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}.$$

T. A. A. B.

1553. *Reflections of an elastic particle.*

The following idealisation is suggested by the motion of a ball on a billiard table.

A particle is projected along the surface of a "billiard table" $ABCD$ from a point in AD distant α from A and after a total of m reflections at AB and CD and a total of n reflections at AD and BC returns to a point in AD distant ω from A. If the angle which the direction of projection makes with DA is θ, it is required to find a relation between θ, α and ω.

Let $AD=a$, $AB=b$, and let the coefficient of restitution (assumed equal for all four reflecting surfaces) be denoted by e. There will be no loss in generality if the initial velocity of projection is taken as unity and $\theta\leqslant 90°$.

On impact the component of the velocity of the particle perpendicular to the surface is reduced in the ratio $e:1$, and it is easy to show that after two successive impacts on adjacent sides of the table the particle is travelling parallel to its former path.

Let the time taken to complete the path of the particle be T. Then
$$T=\frac{\alpha}{\cos\theta}+\frac{a}{e\cos\theta}+\frac{a}{e^2\cos\theta}+\ldots+\frac{a}{e^{m-1}\cos\theta}+\frac{a-\omega}{e^m\cos\theta}\text{ if } m \text{ is even,}$$
or
$$T=\frac{\alpha}{\cos\theta}+\frac{a}{e\cos\theta}+\ldots\ldots\ldots\ldots+\frac{a}{e^{m-1}\cos\theta}+\frac{\omega}{e^m\cos\theta}\text{ if } m \text{ is odd.}$$
Also
$$T=\frac{b}{\sin\theta}+\frac{b}{e\sin\theta}+\ldots\ldots\ldots\ldots+\frac{b}{e^n\sin\theta}\ (n \text{ must be odd}).$$

If m is even, $e^mT\cos\theta=\alpha e^m-\omega+\dfrac{a(1-e^m)}{1-e}$;

and if m is odd, $e^mT\cos\theta=\alpha e^m+\omega+\dfrac{ae(1-e^{m-1})}{1-e}$.

Also
$$e^nT\sin\theta=\frac{b(1-e^{n+1})}{1-e}.$$

Thus if m is even, $e^{n-m}\tan\theta = \dfrac{b(1-e^{n+1})}{a(1-e^m)+(\alpha e^m-\omega)(1-e)}$;

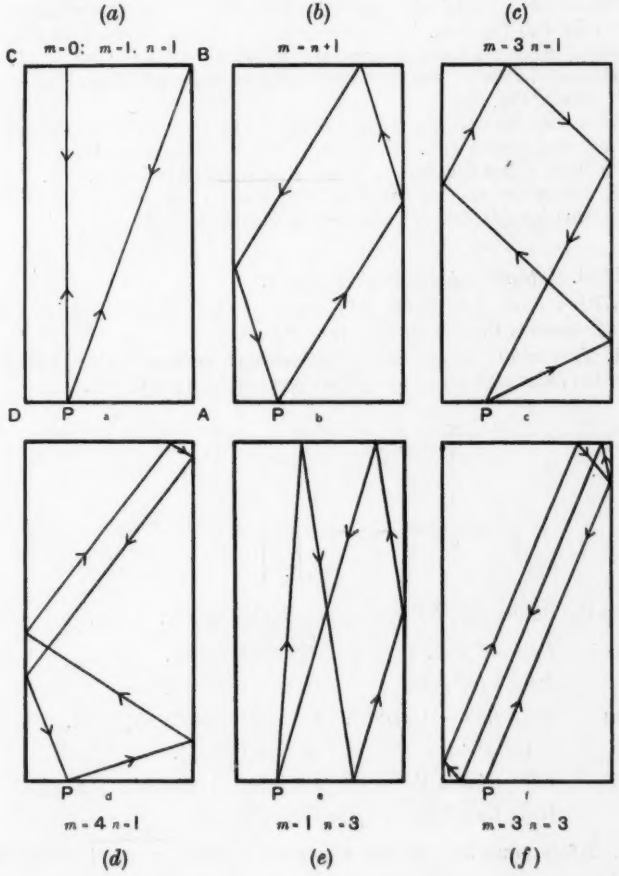
and if m is odd,

$$e^{n-m}\tan\theta = \frac{b(1-e^{n+1})}{ae(1-e^{m-1})+(\alpha e^m+\omega)(1-e)}.$$

To find the angle of projection necessary for the particle to return to the original point of projection put $\omega=\alpha$.

Some special cases in which $\omega=\alpha$ are of interest.

If $m=0$, then for any value of n the path is a straight line parallel to AB (Fig. (a)).



If $m=1$, $n=1$, the angle of projection is arctan (b/a), *i.e.* the particle travels to the corner B whence it returns (Fig. (a)).

The particle will repeat its path if it impinges on AD in a direction making an angle arctan $(e^{-1}\tan\theta)$. The components of the returning speed of the particle are $e^m\cos\theta$ parallel to AD and $e^n\sin\theta$ parallel to AB, and the particle will return to AD making an angle arctan $(e^{n-m}\tan\theta)$ with AD. Thus the particle will repeat its path only if $m=n+1$. Hence in this case m must be even. If $\omega=\alpha$ and $m=n+1$,

$$\tan\theta=eb/\{a-a(1-e)\},$$

which is the relation for the special case $m=2$, $n=1$ (Fig. (b)).

Thus the particle will repeat its path only if, after striking AB, BC and CD in turn, it returns to the point of projection and all cases of $m=n+1$ are simply a repetition of the case $m=2$, $n=1$.

Hence if $m=n+1$ the path will be the same parallelogram repeated $m/2$ times (Fig. (b)).

If $m=n$, the particle returns along a line making an angle arctan $(\tan\theta)$, *i.e.* θ with DA. Thus the particle returns to the point of projection along the line on which it was projected (Fig. (f)).

A few other special cases in which $\alpha=\omega$ are illustrated in the diagrams (c), (d), (e). P denotes the starting-point.

RAYMOND SMART.

1554. *Integral-sided geometric figures.*

With reference to Note 1514, some of the following results may be of interest, though there is nothing new.

1. Trapezoid whose sides, diagonals, perpendicular between parallel sides and area can all be represented by integers.

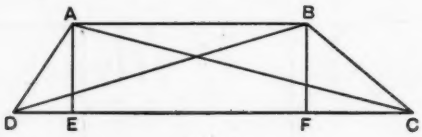


FIG. 1.

In the figure AE, BF are perpendicular to DC.

Let $DE=m^2-n^2$, $AE=BF=2mn$,

$DF=n(m^2-1)$, $CF=m(n^2-1)$;

then $DC=(mn-1)(m+n)$, $EF=AB=(m^2+n)(n-1)$,

$DA=m^2+n^2$, $BC=m(n^2+1)$,

$BD=n(m^2+1)$, $CE=(m+1)(m-n)(n-1)$,

$$AC^2=(m+1)^2(m-n)^2(n-1)^2+4m^2n^2.$$

This latter must be a perfect square in order to make AC rational.

Assuming

$$(m+1)^2(m-n)^2(n-1)^2+4m^2n^2=(m+1)^2\{n^2-(m-1)n+m\}^2$$

and solving for n we get

$$n=m(m^2+m+1)/(m-1)(m+1)^2 \quad \text{where } m>1.$$

Putting $m=2$ and multiplying the values obtained by 81 we get

$$AB=250,\quad BC=554,\quad CD=608,\quad DA=520,$$
$$BD=630,\quad AC=696,\quad AE=504,\quad \text{area}=216216.$$

2. Cyclic quadrilateral whose sides, diagonals, perpendiculars from vertices on diagonals, diameter of circumscribing circle and area are all integers.

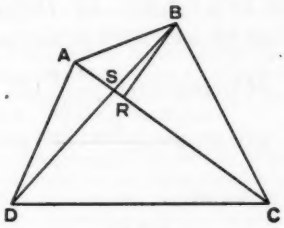


FIG. 2.

BR is drawn perpendicular to AC.

Put $AB=2fg(m^2+n^2)$, $AR=2fg(m^2-n^2)$, then $BR=4fgmn$.

Put $BC=(f^2+g^2)\,.\,2mn$, then $CR=2mn(f^2-g^2)$.

Put $BS=(n^2+g^2)2fm$, then $RS=(n^2-g^2)2fm$.

$$AS=2f(m+g)(mg-n^2),\quad CS=2m(f+n)(fn-g^2).$$
$$DS=(mg-n^2)(m+g)(nf-g^2)(f+n)/(n^2+g^2).$$
$$BD=2fm(n^2+g^2)+\{(mg-n^2)(m+g)(nf-g^2)(f+n)/(n^2+g^2)\}.$$
$$AC=2fg(m^2-n^2)+2mn(f^2-g^2).$$
$$CD=2g(m^2+n^2)(nf-g^2)(f+n)/(n^2+g^2).$$
$$AD=2n(f^2+g^2)(mg-n^2)(m+g)/(n^2+g^2).$$

These give rational but not necessarily integral results.

Consider the case where the diagonals intersect at right angles. Then $g=n$, and we now get

$$AB=2fn(m^2+n^2),\qquad AS=2fn(m^2-n^2),$$
$$BC=(f^2+n^2)2mn,\qquad CS=(f^2-n^2)2mn,$$
$$BS=4fmn^2,\qquad DS=(m^2-n^2)(f^2-n^2),$$
$$BD=4fmn^2+(m^2-n^2)(f^2-n^2),\qquad AC=2n(f+m)(fm-n^2),$$
$$CD=(m^2+n^2)(f^2-n^2),\qquad DA=(f^2+n^2)(m^2-n^2).$$

Area of triangle $ABC = 4fmn^2(f+m)(fm-n^2)$.

Diameter of circumscribing circle of $ABCD = (f^2+n^2)(m^2+n^2)$.

Area of triangle $ADC = n(f^2-n^2)(m^2-n^2)(fm-n^2)$.

Area of $ABCD$ = area of triangle ABC plus area of triangle ADC.

It is also interesting to note that

radius of inscribed circle of triangle $ABC = 2n^2(fm-n^2)$;

radius of inscribed circle of triangle $ADC = n(f-n)(m-n)$.

3. As an alternative to method 2 we can start from the geometrical results:

$$AC^2 = (AD.BC + AB.DC)(AD.AB + BC.CD)/(AB.BC + AD.DC),$$

$$BD^2 = (AD.BC + AB.DC)(AB.BC + AD.DC)/(AD.AB + BC.CD).$$

Let $AB = x, BC = mx, CD = nx, AD = px$, and we have the equations

$$AC^2 = (mp+n)\left(\frac{mn+p}{m+np}\right)x^2,$$

$$BD^2 = (mp+n)\left(\frac{m+np}{mn+p}\right)x^2.$$

Put $(mn+p)/(m+np) = a^2$.

Then $$p = \left(\frac{n-a^2}{a^2n-1}\right)m.$$

AC, BD will be rational if

$$mp+n = \left(\frac{n-a^2}{a^2n-1}\right)m^2+n \text{ is a square.}$$

Let $n = q^2$; then $\left(\frac{q^2-a^2}{a^2q^2-1}\right)m^2+q^2$ must be a square.

Suppose that this expression is $\left(\frac{mr}{a^2q^2-1}+q\right)^2$.

Then $$m = 2qr(a^2q^2-1)/\{(q^2-a^2)(a^2q^2-1)-r^2\}.$$

Put $a = 3/2$, $r = 3/2$, $q = 2$. Then we have

$m = 192/47$, $n = 4$, $p = 42/47$, $AC = 585x/141$, $BD = 260x/141$.

Take $x = 141$; then

$AB = 141$, $BC = 576$, $CD = 564$, $AD = 126$, $AC = 585$, $BD = 260$.

B. J. LEWSLEY.

1373. "—To think that two and two are four
And neither five nor three
The heart of man has long been sore
And long 'tis like to be."

—A. E. Housman, "When First My Way to Fair I Took", printed in *The Century's Poetry*, p. 289. [Per Mr. F. W. Kellaway.]

REVIEWS.

Waves. By C. A. COULSON. Pp. xii, 156. 5s. 1941. (Oliver and Boyd)

The headings of the eight chapters are as follows. The equation of wave motion, waves on strings, waves in membranes, longitudinal waves in bars and springs, waves in liquids, sound waves, electric waves, general considerations.

The author's object is to consider from an elementary standpoint as many different types of wave motion as possible. The mathematical problem of waves is substantially the solution of the equation of wave motion :

$$c^2\nabla^2\phi=\frac{\partial^2\phi}{\partial t^2},$$

subject to assigned boundary conditions; the physical problem lies in interpreting ϕ and the properties of the solutions. From this point of view there is much to be said for a unifying discussion of the type here presented, for it brings within the reach of the general mathematical reader a variety of topics which he might otherwise miss, and it forms a valuable corrective to the ideas of the future specialist who might otherwise take a too myopic view of the particular branch which he will study in detail.

If a small criticism may be permitted one feels that the author has laid too little emphasis on the fundamentally important notion of group velocity. The opening sentence deals with the pebble dropped into a pond and it is here that group velocity enters immediately, for the visible waves are due to the reinforcement of members of groups of microscopic amplitudes arriving in the same phase. The group velocity dn/dm follows at once from this observation. It is therefore rather misleading to regard group velocity as an exceptional phenomenon arising from two waves of equal amplitudes (p. 130).

Each chapter has an excellent collection of exercises for the reader. The book forms a worthy member of the series of " University Mathematical Texts ". L. M. M.-T.

The Calculus of Extension. By H. G. FORDER. Pp. xvi, 490. 30s. 1941. (Cambridge)

The late Professor Genese, on his death in 1928, left to the Mathematical Association his lecture notes and collections of examples on the methods of the Grassmann calculus. These passed into the hands of Professor Forder, and eventually provided the inspiration of the present book. The main object of this work is to demonstrate the power of Grassmann's methods when applied to Geometry. Into this task the author has put unlimited care and devotion, and he has every reason to be satisfied with the result of his labours.

When a book is written with the sole object of illustrating the power of a particular method applied to well-known topics, and when this book extends to nearly 500 pages, the reader is apt to finish up either with unlimited enthusiasm for the method, or with the reverse. Usually, either of these results is unfair to the method. If the method is a good one, it should have its place in the literature of the subject to which it is applied, but seldom does it displace any other tried method. Professor Forder has proved himself a good advocate for Grassmann's methods, and the numerous applications which he has made show how capable these methods are of providing elegant proofs of well-known theorems. He has, indeed, established their claim to a place among the geometer's armaments, and many may consider that he has done much more than this. Whether there are many who will adopt them in their search for new results is perhaps not as certain; considerable practice will

first be necessary, but there seems no obvious reason why eventually the methods described in this book should not prove potent in the hands of researchers.

Perhaps the main difficulty arises from the simultaneous use of two types of multiplication, progressive and retrogressive, which results in multiplication not always having the associative property. The difficulty so caused is apparent rather than real, for the multiplication which is not associative is merely an abbreviated notation for associative multiplication together with a second operation; but the representation by non-associative multiplication often makes it necessary to prove formulae which have the appearance of *ad hoc* results, thus giving an air of difficulty to the proofs of certain well-known theorems.

Much weight is placed on the fact that the elements manipulated are geometrical entities, and not merely coordinates. This point is a very good one, though too much can be made of it, for in this the author's method does not really differ in any essential respect from other symbolic methods which are based on coordinate systems. The difference is more a matter of emphasis. Something is gained by the introduction of additional entities such as vectors, rotors and bivectors, but at the expense of some complication of ideas.

The book will impress different readers in different ways. It is a book to browse over, rather than to plod through steadily. One should not try to base a whole course of lectures on it, for a one-sided view of the subject would result. But undoubtedly a great many will derive much pleasure as well as profit from picking it up from time to time, and we can safely predict that many will find it a valuable addition to their bookshelves. W. V. D. H.

British Association for the Advancement of Science. Mathematical tables. VIII. Number-divisor tables. Pp. x, 100. 15s. 1940.

IX. Tables of powers giving integral powers of integers. Pp. xii, 132. 15s. 1940. (Cambridge)

The first of these volumes was designed and in part prepared by J. W. L. Glaisher, and extended and edited by the British Association Committee for the calculation of mathematical tables.

The second was initiated by J. W. L. Glaisher and extended on behalf of the committee by W. G. Bickley, C. E. Gwyther, J. C. P. Miller, E. J. Ternouth.

The publication of both was made possible by Lieut.-Col. Allan Cunningham's bequest to the British Association. Unquestionably, he would have rejoiced in the use made of his bequest in publishing these tables. They are characteristic of what has come to be expected from the Association tables: clear and fair-sized print and pages; well-designed and conveniently arranged tables, especially the second, where much thought must have been required; easy for reference; and with every conceivable precaution to ensure accuracy.

Vol. VIII is primarily a table for arithmeticians. Denote by $\phi(n)$ the number of positive integers prime to and not exceeding n, by $\nu(n)$ the number of divisors of n including 1 and n ($d(n)$ is the modern notation), and by $\sigma(n)$ the sum of all the divisors of n. The tables contain: for $n \leqslant 10{,}000$ the factors of n and the values of these functions; the solutions of the equation $\phi(x)=m$ for $m \leqslant 2500$; the solutions of $\nu(x)=m$ for $m \leqslant 64$ and $x \leqslant 10{,}000$, those for which the Moebius function $\mu(x)$ is not zero being specially indicated; and finally the solutions of $\sigma(x)=m$ for $m \leqslant 10{,}000$.

This volume will be invaluable for those requiring arithmetical data concerning the functions tabulated. It should prove of great assistance in leading to theoretical results, especially as there are a number of problems still outstanding for the functions above. Thus the tables verify for $m \leqslant 2500$ the

conjecture that when $\phi(x)=m$ is solvable for given $m>1$, there are at least two solutions, a result erroneously proved by Carmichael.

The value of the tables is enhanced by the interesting introductions containing historical and technical matter and full references to previous tables. For Vol. VIII, these are due to D. H. Lehmer and J. Wishart, and for Vol. IX to J. C. P. Miller.

Vol. IX contains the powers x^n for $x\leqslant 1099$, $n\leqslant 12$; $x\leqslant 299$, $n\leqslant 20$; $x\leqslant 120$, $n\leqslant 50$. Since the tables are designed both for research in number-theory, and for general computation where only a limited number of leading digits is required, a compromise has been effected between the two points of view. For powers above the 30th, only the 40th and 50th are printed in full, and for $x>100$ the 28th and 29th powers are also curtailed. The volume concludes with a short table for evaluating sums of powers and contains the Bernoulli numbers up to B_{99} in their lowest terms. Except for the last part, the tables have obviously a strong utilitarian character and are indispensable for those requiring powers. L. J. MORDELL.

The Mathematical Work of John Wallis. By J. F. SCOTT. Pp. xi, 240. 12s. 6d. 1938. (Taylor & Francis)

John Wallis (1616-1703) occupies an important position in the early history of the Calculus. He attempts to detach the idea of the integral from the conception of an area and to apply to it, if possible, some of the new methods of algebra, *i.e.* he tries to translate Cavalieri's atomising of the continuum into a rigorous arithmetical "limes"-operation. He did not succeed in any general way, in default of a powerful symbolism; but by his ingenious method of "interpolation" (nothing less than Analytic Continuation!) he obtained astonishing results and so helped essentially to induce further work aiming at this abstraction and generalisation which was to culminate in Newton's Calculus.

Dr. Scott gives us the story of Wallis' education and early academic life, in which mathematics does not play any great part before his thirtieth year; then his powers quickly developed, and in 1649 he became Savilian professor of Geometry in Oxford. He wrote on the new Analytical Geometry, he solved a great many problems of integrations by his own method which led him to the discovery of the infinite product for π; he collected and summed up all that had been done in Algebra in a big treatise which became a standard work for a long time. He was in correspondence with all the foremost scientists of his time—frequently involved in disputes on "nationalistic" claims of priority. He wrote an important book on Mechanics and he edited ancient mathematical texts from manuscripts. Apart from this work in natural science he became famous as the author of a grammatical system of the English language, and he showed exceptional skill in deciphering secret codes. This was of considerable importance on some occasions during the Civil War, but he also made use of it, less seriously perhaps, to cheat Huygens in the matter of the system of Saturn, by composing an anagram from which he was able, after Huygens' discovery, to deduce anything he liked, in particular a statement which seemed to show that he had discovered this system before Huygens: he confessed, however, what had happened, and the correspondence between the two went on for almost twenty years after this "joke". Dr. Scott does not only give us detailed and clear analyses of the main works: *Arithmetica Infinitorum*, *Mechanica* and *Algebra*; but he succeeds by extensive quotations from MS. material (when will it be published?) and of much contemporary literature to make his story extremely lively and real. We meet all the great men of the period, are able to observe the manners of their intercourse, within England and across the sea—and we may well wonder

how very Nazi-like their arguments became when they grew hot in their disputes. Dr. Scott says: "Mathematics is so fundamental and so abstract that it reaches to the very foundations of thought, and seems to provoke the most unaccountable reactions in the minds of great thinkers. . . ."

The book ends with a highly useful (even if not quite reliable) catalogue of contemporary scientists (Schooten, the son, lived 1615-60; Anderson did not know Vieta personally; Kepler's laws are in the *Astr. Nova*). Another welcome appendix gives reproductions of some pages of the works of Wallis and others, to give an idea of their notation; the preceding essay, however, is too scanty to be of any value in this form.

One would like to see the mathematical analysis carried one step further to give (in Wallis' own words) "the true anatomy" of the inherent structure so that one might see, in modern form, the actual mathematical connection, where and why a proof fails, what material is required for the success of some deduction; this search "behind" the facts in their historical guise is important, I believe, not only to show the depth of our author's reasoning and insight, but it also gives frequently the only means to take sides in one of the various quarrels. (Such an analysis, to some extent, but *without* the original text, was given for the product for π in this *Gazette*, May 1937, by Dickinson.)

There are a number of misprints and obvious slips, a list of which will be sent to the author; I note here only some of the major points where I think Dr. Scott is not quite correct: (1) Wallis' share in the rectification-problem is overrated; the significant device of Heuraet's (general) and Neil's (special) construction consists in the transformation into a quadrature by means of the infinitesimal triangle; neither Neil nor Heuraet adopt Wallis' method of integration by "arithmetica infinitorum". Roberval's and Wren's (different) methods for the arc of the cycloid appeared less important as being not related to algebraic curves. (2) Wallis' graphical representation of complex numbers is correct and consistent, and general enough if one constant is made equal to 1. Our number $u+iv$ is given by a point x, y where $x=u-1\sin\phi$, $y=1\dfrac{\sin^2\phi}{\cos\phi}$ with $v/1=\tan\phi$, $0\leqslant|\phi|\leqslant\frac{1}{2}\pi$; $u-iv$ is given by $x'=u+1\sin\phi$, $y'=y$. (3) Wallis was not outstanding in asking for "general" methods—so did Cavalieri, Roberval, Descartes, Fermat, Gregory, Barrow, and others. Similarly (at the beginning of the chapter on Mechanics) it should be said that Wallis *like* (not at all "unlike") most of his predecessors and contemporaries did not restrict his work to pure mathematics: see Torricelli, Pascal, Roberval, Descartes, Gregory—not to mention Kepler and Huygens; Mechanics was not as neglected as Dr. Scott will have us believe. And (4) to defend historians of mathematics against the reproach (p. 1) of having neglected Wallis too much, here are some papers: (*a*) Wertheim, *Abh. zur Gesch. d. Math.*, ix, 1899, on the *Commercium Epistolicum*; (*b*) Wieleitner, *Weltall*, 29, 1930, on the *Conic Sections*; and (*c*) in *Quell. und Stud. z. Gesch. d. Math.*, B, i, 1930, appeared a rather comprehensive paper on the whole of Wallis' work.

The specialist will probably not quite absolve Dr. Scott's book from the blame which Prof. Andrade expresses (with respect to other works) in the first sentence of his foreword; this applies particularly to all places where general statements are made: but this must not divert us from the merits of this book in giving a richly documented and very readable account of one of the great figures of the Heroic Age of Mathematics. One could hardly think of a better choice than Wallis' life and work to show the growth of scientific ideas and ideals in times when men did not find it easy to do scientific research, and had to make rules that "barred all discurses of State affairs and of News (other than what concerned the business of Philosophy)". . . A. P.

A mathematical prodigy : history and legend. By C. J. KEYSER. Reprinted from *Scripta Mathematica*, V, 2. Pp. 12. 20 cents. 1938. (New York)

The story of Pascal's life and works has frequently been told, but never, I believe, in such a dramatic and fascinating way as in this article. The spiritual and social background of Pascal's epoch is depicted by a few significant episodes in the hero's life, and Keyser leads us to accept as highly credible the " legend " of how the boy Pascal discovered Mathematics. (The present reviewer has once based a first course in geometry on this story : to let children " play " with " rounds " and " bars " ; but then a whole class united their efforts!) Keyser surveys briefly all achievements of Pascal in science, philosophy and literature, and he ends by emphasising the enigmatic character that Pascal still presents to posterity—and the tragic struggle of so " sensitive a soul ". Papers like this give a welcome introduction into the History of Mathematics and are well within the grasp of boys' interests. A. P.

Greek mathematics and astronomy. By Sir THOMAS HEATH. Reprinted from *Scripta Mathematica*. Pp. 215-232. 25 cents. 1938. (New York)

The Greek approach to mathematics will in some way always remain the " real " beginning of the story for us, because apparently we can revive in our minds their quest for logical proof and their urge to speculate about structure and reason of the world. In this article Heath himself gives a concise survey in 18 pages, with particular care to show the development before the Golden Age of Euclid, Archimedes and Apollonius, which is all the more valuable as it comes from the author whose *History of Greek Mathematics* has become the classic work on the subject. A. P.

Outline of the History of Mathematics. By R. C. ARCHIBALD. 4th ed. Pp. 66. 1939. 50 cents. (Mathematical Association of America, Oberlin, Ohio)

The fact that a fourth edition of this pamphlet had to be brought out shows how welcome such a condensed survey is as a source of easy and quick information ; and this " Outline " attempts also, by an extensive and up-to-date list of books and papers (chiefly in English), to enable the reader to do his own studies. But since this little work is intended for the wider public, such studies are not guided to the ancient texts themselves—although it may seem to some of us that it is one of the first necessities for the beginning student of History of Mathematics to read an old author, Greek or seventeenth century.

The new edition has been enriched by more details about pre-Greek mathematics which have been made available by Neugebauer's discoveries during the last years. Most of the further changes have been relegated into the " Notes ", and it is now difficult in some instances to obtain a fair view of the relative importance of some mathematicians ; *e.g.* Barrow. (In the case of Kepler the author apologises for this " injustice ".)

As is necessary in such a work intended as a guide for quick reference, the accuracy of the given facts is very high. It may be permitted, however, to mention a few slips : amongst sixteenth century algebraists the German Stifel should have been mentioned (who, *e.g.*, had " Pascal's triangle ") ; Desargues was rather an outsider, not a " member of the French school " ; nothing is said about Pascal's contribution to the early calculus ; Wallis' effort to represent an imaginary number graphically is different from the method now adopted (but complete and consistent in itself) ; and the rectification of a curve should not be booked to his credit, but to Heuraet's and Neil's.

A. P.

Geschichte der Mathematik. By H. WIELEITNER. New impression. I. Von den ältesten Zeiten bis zur Wende des 17. Jahrhunderts. Pp. 136. Rm. 1.62. II. Von 1700 bis zur Mitte des 19. Jahrhunderts. Pp. 154. Rm. 1.62. 1939. Sammlung Göschen, 226, 875. (Walter de Gruyter, Berlin)

In these two small volumes (130 and 150 pp. 8vo) Wieleitner accomplished a most astonishing task. He has indeed given a full account of the whole of the History of Mathematics from the earliest times up to the death of Gauss. The great value of this work consists in its reliability as a book of reference. Although Wieleitner's main work in history was concentrated on sixteenth and seventeenth century mathematics, he did original research in many other periods, and wherever one opens this little compendium one can find proof of this in the amazing wealth of accurate data.

The new edition is, apart from a second register in the second volume, an exact reprint of the earlier edition; there is consequently no reference to recent discoveries, *e.g.* in Babylonian mathematics. There seems to be no book equivalent to this in English, and one might wish that students of the History of Mathematics in this country would use this small work to avoid a permanent repetition of errors by quoting from older books that have not this standard of reliability! A. P.

Introduction to Astronomy. By R. H. BAKER. 2nd edition. Pp. 315. 12s. 6d. 1940. (Macmillan)

This introduction to the study of astronomy is in essence an abridged edition of the author's larger volume *Astronomy*, which received notice in these pages some months ago. *Introduction to Astronomy* makes very complete its purpose " to tell the story of the heavens in a way that will be understandable without special preparation ". It makes no attempt, therefore, to start the student on his way to solving little problems involving spherical astronomy. Perhaps the author is right in thus luring on the student by first giving him a bird's-eye view of the range of astronomical phenomena from this mundane atmosphere capturing the celestial meteoric visitor to the galaxies in remotest space. No one can remain insensitive to the tale unfolded by text, diagram and photograph.

This second edition of the *Introduction*, first published in 1935, and intended primarily to suit the needs of American students, includes references to the 200-inch reflector telescope which is nearing completion, to recently discovered satellites of Jupiter, the cinematography of solar prominences and further considerations on the origin of meteors. The subject-matter is clearly set out under bold headings, and the illustrations, chiefly from American sources, are excellent. H. W. N.

1374. When I was a small boy my first night's home-work consisted of the first proposition in the first book of Euclid. I did not gather that the letters *A*, *B*, and *C* were mere labels having no property except to identify certain geometrical relationships. I thought they had existence in themselves. Not knowing what Euclid was about, I made no effort to relate the letters to the " illustration ", and the only way, as I thought, to master the proposition was to get it by rote. Next day I went to the blackboard and reeled off what I had learned. Perceiving what lay behind so much glibness, the master said: " Never mind *A*, *B*, and *C*. Call them *X*, *Y*, and *Z*." Whereupon I was utterly flummoxed.—From James Agate's notice in the *Sunday Times* (Nov. 27th, 1938) of the Old Vic production of *Man and Superman* by G. B. Shaw. [Per Mr. P. J. Harris.]

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